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Thin domains with non-smooth periodic oscillatory boundaries ☆

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A B S T R A C T

In this work we study in detail how to adapt the unfolding operator method to thin domains with periodic oscillatory boundaries. We present the unfolding method as a general approach which allows us to analyze the behavior of the solutions of a Neumann problem for equation $-\Delta u+u=f$ posed in two-dimensional thin domains with an oscillatory boundary. Assuming very mild hypothesis on the regularity of the oscillatory boundary we obtain the homogenized limit problem and corrector results for the three different cases depending on the order of the period of the oscillations.

1. Introduction

In this paper we provide a comprehensive presentation of the periodic unfolding method for thin domains with a periodic oscillatory boundary. We adapt the unfolding method introduced in [17] by D. Cioranescu, D. Damlamian and G. Griso (see also [18]) to thin domains with an oscillatory boundary and show that it provides a general approach to analyze in a systematic and unified way, certain elliptic problems posed in this kind of domains.

Throughout this paper, we consider thin domains with order of thickness $\epsilon > 0$ which are defined as follows

$$R^\epsilon = \Big\{(x, y) \in \mathbb{R}^2 \mid x \in (0,1),\ 0 < y < \epsilon\, g(x/\epsilon^\alpha)\Big\}, \tag{1.1}$$

where $\alpha > 0$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is an L-periodic function not necessarily smooth (see hypothesis $(\mathbf{H_g})$ in Section 2) satisfying $0 \le g_0 \le g(\cdot) \le g_1$ for some fixed non-negative constants g_0 and g_1. Moreover, in the

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whole paper we will assume that the function g is a nonconstant function in order to guarantee the oscillating behavior of the upper boundary.

Observe that the different values of $\alpha > 0$ will give us different types of oscillatory behavior or rugosity at the boundary. We will distinguish three different cases taking into account the relation between the order of the thickness of the domain and the order of the period of the oscillations. More precisely:

- $0 < \alpha < 1$. We will refer to this case as "weak oscillatory" case. The order of the period of the oscillations is ϵ^α, which is much larger than the order of the amplitude of the oscillations, ϵ, or the order of the thickness of the domain, also ϵ. Notice that in this case, if the function g is smooth enough, then the function $x \to \epsilon g(x/\epsilon^\alpha)$ is uniformly $C^{1,\theta}$ for any $\theta < 1 - \alpha$ and it goes to zero in $C^{1,\theta}(\mathbb{R})$.
- $\alpha = 1$. We will refer to this case as "resonant" or "critical" case. Notice that the order of the period coincides with the order of the amplitude of the oscillations and it also coincides with the order of the thickness of the domain. Moreover, if again the function g is smooth enough, then the function $x \to \epsilon g(x/\epsilon)$ is uniformly C^1 but it does not go to 0 in this topology.
- $\alpha > 1$. We will refer to this case as "fast" or "extremely high oscillatory" case. The order of the period of the oscillations is much smaller than the order of the amplitude of the oscillations or the order of the thickness of the domain. The function $x \to \epsilon g(x/\epsilon^\alpha)$ is uniformly bounded in some Hölder norm but not in C^1.

Assuming that R^ϵ is connected, $g_0 > 0$, we study the behavior of the solutions of the following Neumann problem

$$\begin{cases} -\Delta u^\epsilon + u^\epsilon = f^\epsilon & \text{in } R^\epsilon \\ \dfrac{\partial u^\epsilon}{\partial \nu^\epsilon} = 0 & \text{on } \partial R^\epsilon \end{cases} \tag{1.2}$$

where $f^\epsilon \in L^2(R^\epsilon)$ and ν^ϵ is the unit outward normal to ∂R^ϵ. The variational formulation is: find $u^\epsilon \in H^1(R^\epsilon)$ such that

$$\int_{R^\epsilon} \left\{ \frac{\partial u^\epsilon}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u^\epsilon}{\partial y}\frac{\partial \varphi}{\partial y} + u^\epsilon \varphi \right\} dxdy = \int_{R^\epsilon} f^\epsilon \varphi \, dxdy, \quad \forall \varphi \in H^1(R^\epsilon). \tag{1.3}$$

Observe that, the existence and uniqueness of solutions for problem (1.3) are guaranteed by Lax–Milgram Theorem for every fixed $\epsilon > 0$. Notice also that the behavior of the solutions will depend essentially on the value of the parameter α. Moreover, since the domain R^ϵ has order of thickness ϵ it is expected that the family of solutions u^ϵ will converge to a function of just one variable as the parameter ϵ tends to zero.

Hence, the purpose of this paper is to introduce an unfolding operator which allows us to obtain in an easy way the homogenized limit problem and a corrector result for problem (1.2) in the three different situations described above: weak, critical and fast oscillations. Moreover, the interest of this method comes from the fact that for the homogeneous Neumann problem (1.2) we may admit non-smooth periodic oscillatory boundaries.

Let us point out that there are several papers addressing the problem of studying the effect of rough boundaries on the behavior of the solution of partial differential equations posed in thin domains. We will mention some of them here and we also refer to their corresponding bibliographies. In [25–27] the authors study the asymptotic behavior of solutions to certain elliptic and parabolic problems in a thin perforated domain with rapidly varying thickness. The results obtained in these papers are related to the construction of a suitable asymptotic expansion of the solutions which was proposed by T. Melnyk in [24] for the investigation of elliptic and spectral problems in thin perforated domains with rapidly varying thickness.