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On radial solutions for singular combined superlinear elliptic systems on annular domains

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ABSTRACT

We prove the existence of a large positive solution to the system

$$\begin{cases} -(r^{N-1}\phi_1(u'))' = \lambda r^{N-1} f_1(v), \quad a < r < b, \\ -(r^{N-1}\phi_2(v'))' = \lambda r^{N-1} f_2(u), \quad a < r < b, \\ \quad u(a) = 0 = u(b), v(a) = 0 = v(b), \end{cases}$$

where $a > 0$, λ is a small positive parameter, $f_i : (0, \infty) \to \mathbb{R}$ are continuous and $\lim_{z\to\infty} \frac{\phi_1^{-1}(f_1(c(\phi_2^{-1}(f_2(z)))}{z} = \infty$ for all $c > 0$.

1. Introduction

In this paper we study the existence of positive radial solutions for the quasilinear elliptic system

$$\begin{cases} -\operatorname{div}(\alpha_1(|\nabla u|^2)\nabla u) = \lambda f_1(v), \quad a < |x| < b, \\ -\operatorname{div}(\alpha_2(|\nabla v|^2)\nabla v) = \lambda f_2(u), \quad a < |x| < b, \\ \quad u = v = 0, \ |x| \in \{a, b\}, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^N$, $a > 0$, $\phi_i(s) = \alpha_i(s^2)s$ are increasing homeomorphism on $\mathbb{R}$, $i = 1, 2$, and λ is a positive parameter. Thus we consider the ODE system

$$\begin{cases} -(r^{N-1}\phi_1(u'))' = \lambda r^{N-1} f_1(v), \ a < r < b, \\ -(r^{N-1}\phi_2(v'))' = \lambda r^{N-1} f_2(u), \ a < r < b, \\ \quad u(a) = 0 = u(b), \quad v(a) = 0 = v(b). \end{cases} \tag{1.2}$$

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We make the following assumptions:

(A1) ϕ_i are odd, increasing homeomorphism on $\mathbb{R}$ and for each $c > 0$, there exists a constant $A_c > 0$ with $A_c \to \infty$ as $c \to \infty$ such that

$$\phi_i^{-1}(cz) \geq A_c \phi_i^{-1}(z)$$

for $z \geq 0$ and $i = 1, 2$,

(A2) $f_i : (0, \infty) \to \mathbb{R}$ are continuous and $\lim_{z\to\infty} f_i(z) = \infty$ for $i = 1, 2$.

(A3) $\lim_{z\to\infty} \frac{\phi_1^{-1}(f_1(c(\phi_2^{-1}(f_2(z)))}{z} = \infty$ for all $c > 0$.

(A4) There exists a continuous decreasing function $h : (0, \infty) \to (0, \infty)$ such that $h \in L^1(0, T)$ for all $T > 0$ and

$$\limsup_{z\to 0^+} \frac{|f_i(z)|}{h(z)} < \infty$$

for $i = 1, 2$.

Remark 1.1. (i) Note that (A1) implies $\phi_i^{-1}(cz) \leq B_c \phi_i^{-1}(z)$ for $z \geq 0$, $i = 1, 2$, where $B_c \to 0$ as $c \to 0^+$, while (A3) gives $\lim_{z\to\infty} \frac{\phi_2^{-1}(f_2(c(\phi_1^{-1}(f_1(z)))}{z} = \infty$ for all $c > 0$.

(ii) If $\phi_i(z) = z$ then (A3) reads $\lim_{z\to\infty} \frac{f_1(cf_2(z))}{z} = \infty$ for all $c > 0$.

Our main result is

Theorem 1.1. *Let (A1)–(A4) hold. Then there exists a constant $\lambda_0 > 0$ such that for $\lambda < \lambda_0$, the system (1.2) has a positive solution (u_λ, v_λ) with*

$$\min(u_\lambda(r), v_\lambda(r)) \to \infty$$

as $\lambda \to 0^+$ uniformly on compact subsets of (a, b).

The case when $\phi_i(s) = |s|^{p_i-2}s$, $p_i > 1$, corresponds to the p_i-Laplacian and has been studied extensively in recent years. We are motivated by the result in [16, Theorem 1.1], in which the existence of a positive solution to (1.2) was established for $\lambda > 0$ small when $\phi_1(s) = \phi_2(s) = |s|^{p-2}s$, $p > 1$, $\phi_i^{-1} \circ f_i$ are nonsingular and superlinear at ∞ and when $f_i(0) < 0$ for $i = 1, 2$. This case when the reaction terms are negative at the origin are referred in the literature as semipositone problems and it is well known that the analysis of positive solutions for such problems are very challenging. In this paper, we shall extend the result in [16] to the case which allows a combined superlinear condition at ∞ (one of the $\phi_i^{-1} \circ f_i$ could be sublinear or linear at ∞) and with no sign assumptions imposed on the reaction terms at the origin (also include the singular cases $\lim_{s\to 0^+} f_i(s) = \pm\infty$ for $i = 1, 2$). In particular, in the model case when $f_i(s) = \frac{a_i}{s^{\beta_i}} + b_i s^{q_i}$, where $a_i \in \mathbb{R}$, $b_i > 0$, and $\beta_i \in [0, 1)$, our result only requires $q_1 q_2 > (p_1 - 1)(p_2 - 1)$ instead of the stronger condition $q_i > p_i - 1$ for $i = 1, 2$ in [16]. Existence results for semipositone superlinear systems in a ball were obtained in [11] for $p = 2$ while results on a general bounded domains can be found in [7] and [8] in the cases $p = 2$ and $1 < p < 2$ respectively. However in these references each of the reaction terms is required to satisfy the p-superlinear condition at ∞ and be nonsingular and negative at the origin. To our knowledge this is the first paper analysing systems when the reaction terms satisfy a combined superlinear condition at ∞. For results in the superlinear single equation case, see [1–6,9,13,15,17] and the references therein. Finally we refer to [12,14] for existence results on semipositone systems for λ large when the reaction terms satisfy a combined sublinear condition at ∞.