J. Math. Anal. Appl. 446 (2017) 426–435

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Banach spaces from a construction scheme

Fulgencio Lopez

Department of Mathematics, University of Toronto, Bahen Center 40 St. George St., Toronto, Ontario M5S 2E4, Canada

ARTICLE INFO

Article history:
Received 5 December 2015
Available online 6 September 2016
Submitted by T. Domínguez Benavides

Keywords:
Construction schemes
Almost biorthogonal systems
Uncountable basic sequence
Mazur intersection property

ABSTRACT

We construct a Banach space $\mathcal{X}_\varepsilon$ with an uncountable ε-biorthogonal system but no uncountable τ-biorthogonal system for $\tau < \varepsilon(1+\varepsilon)^{-1}$. In particular the space has no uncountable biorthogonal system. We also construct a Banach space $\mathcal{X}_K$ with an uncountable K-basic sequence but no uncountable K'-basic sequence, for $1 \leq K' < K$. A common feature of these examples is that they are both constructed by recursive amalgamations using a single construction scheme.

1. Introduction

The class of nonseparable Banach spaces exhibit phenomena which are not present in the more studied class of separable Banach spaces. Some of the most striking differences were discovered recently by J. Lopez-Abad and S. Todorcevic [5] when they were developing forcing constructions of Banach spaces via finite-dimensional approximations. For example, it is shown in [5] that for every $\varepsilon > 0$ rational, there is a forcing notion $\mathbb{P}_\varepsilon$ which forces a Banach space $\mathcal{Y}_\varepsilon$ with an uncountable ε-biorthogonal system and such that for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$, $\mathcal{Y}_\varepsilon$ has no uncountable τ-biorthogonal system. They also showed [5, Theorem 6.4] that for every constant $K > 1$ there is a forcing notion $\mathbb{P}_K$ which forces a Banach space $\mathcal{Y}_K$ with an uncountable K-basis yet for every $1 \leq K' < K$, $\mathcal{Y}_K$ has no uncountable K'-basic sequences. Recall that none of these two phenomena can happen in the class of separable Banach spaces when, of course, we replace 'uncountable' by 'infinite'.

In [9], S. Todorcevic introduced a notion of construction scheme of uncountable mathematical objects via finite approximation and simultaneous multiple amalgamations. While the construction scheme $\mathcal{F}$ can be described relying only on ordinary axioms of set theory, their crucial properties of 'capturing' (see the definition below) can only be provided using Jensen's combinatorial principle $\diamondsuit$: there are sets $A_\alpha \subset \alpha$ for every $\alpha < \omega_1$ such that for every A subset of ω_1 there are stationarily many α's with $A \cap \alpha = A_\alpha$.

E-mail address: fulgencio.lopez@mail.utoronto.ca.

http://dx.doi.org/10.1016/j.jmaa.2016.08.068

The purpose of this note is to apply this construction scheme to the theory of nonseparable Banach spaces inspired by the forcing constructions of [5]. In particular, we prove the following two results.

Theorem 1. *Assume $\diamondsuit$. Then for every $\varepsilon \in (0,1) \cap \mathbb{Q}$, there is a Banach space $\mathcal{X}_\varepsilon$ with an uncountable ε-biorthogonal system but no uncountable τ-biorthogonal system for every $0 \leq \tau < \frac{\varepsilon}{1+\varepsilon}$.*

Theorem 2. *Assume $\diamondsuit$. Then for every constant $K > 1$, there is a Banach space $\mathcal{X}_K$ with a K-basis of length ω_1 but no uncountable K'-basic sequence for every $1 \leq K' < K$.*

In each case the construction is based on a single rule of multiple amalgamation of a family of finite-dimensional Banach spaces indexed by $\mathcal{F}$. This adds not only to the clarity over the corresponding forcing constructions but it also gives us Banach spaces that could be further easily analyzed. In fact neither the construction nor the analysis of the corresponding examples require any expertise outside the Banach space geometry.

It is interesting to compare our examples with the corresponding examples in [5]. Given an uncountable sequence of forcing conditions, take an uncountable Δ-subsequence where all conditions are isomorphic and find a condition which amalgamates finitely many of these forcing the desired inequality. Thus, the use of forcing allows us to amalgamate a posteriori since the generic filter G takes care of all the possible Δ-systems whose roots belong to G. However in our recursive construction the amalgamations must be done a priori which limits the class of possible amalgamations. In fact since we do a single amalgamation at any given level of $\mathcal{F}$, our spaces tend to be considerably more homogeneous and therefore much easier to analyze.

In Section 3 we give a proof of Theorem 1 and in Section 4 we prove Theorem 2.

2. Preliminaries

We use standard notation for set theory. For $\alpha \in \omega_1$ we denote the αth ordinal and the set $\{\beta : \beta < \alpha\}$. If $A, B \subset \omega_1$ we say $A < B$ if for all $a \in A$ and $b \in B$, $a < b$.

We follow standard notation for Banach spaces (see, for example, [4] and [2]). In particular $c_{00}(\omega_1)$ is the vector space of functions $x : \omega_1 \to \mathbb{R}$ with finite support (we use $\mathrm{supp}(x)$ for the support of x). If F is a finite subset of ω_1 and $h : F \to \mathbb{R}$, we consider the extension of h in $c_{00}(\omega_1)$ to be zero outside of F and still refer to it as h without risk of confusion. By e_α we denote the function on ω_1 that takes α to 1 and every other $\beta \in \omega_1$ to zero. For approximation purposes we work most of the time on $c_{00}(\omega_1, \mathbb{Q})$, meaning we consider functions in $c_{00}(\omega_1)$ that only take values in $\mathbb{Q}$.

If $h, x \in c_{00}(\omega_1)$ we denote

$$\langle h, x \rangle = \sum_{\alpha < \omega_1} h(\alpha) x(\alpha)$$

which is well defined because x and h have finite support.

We recall some notions of Banach space theory relevant for the results.

Definition 3. Let $\mathcal{X}$ be a Banach space and $(y_\alpha, y^*_\alpha)_{\alpha<\omega_1}$ a sequence in $\mathcal{X} \times \mathcal{X}^*$. For $\varepsilon \geq 0$, we say that $(y_\alpha, y^*_\alpha)_{\alpha<\omega_1}$ forms an *ε-biorthogonal system* if $y^*_\alpha(y_\alpha) = 1$ for every $\alpha < \omega_1$, and $|y^*_\alpha(y_\beta)| \leq \varepsilon$ for every $\alpha \neq \beta$. If $\varepsilon = 0$ we say $(y_\alpha)_{\alpha<\omega_1}$ forms a *biorthogonal system*.

A Banach space $\mathcal{X}$ has the *Mazur intersection property* (MIP) if every closed convex subset of $\mathcal{X}$ is the intersection of closed balls.