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Analysis and numerical simulation of the three-dimensional Cauchy problem for quasi-linear elliptic equations

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A B S T R A C T

This work is concerned with solving the Cauchy problem for quasilinear elliptic equations whose exponential instability is manifestly seen by the catastrophic growth in the representation of the exact solution. Our proposed regularization procedure consists in damping the unbounded terms in the representation. Moreover, we show that its solution converges to the exact solution uniformly and strongly in L^2 under a priori assumptions on the exact solution. In order to verify our analysis and the accuracy of the numerical procedures, we exhibit two numerical examples. Our main tools for simulation are the trigonometric polynomial approximation, and the fast Fourier transform in combination with the cubic Hermite interpolation.

1. Introduction

It is well known that the Cauchy problem for second-order elliptic partial differential equations is ill-posed in the sense of Hadamard [5], i.e., the solution does not exist, or even it exists, it does not depend continuously on the Cauchy data. An answer to the ill-posedness issue of such Cauchy problem can be found in Faker Bin Belgacem et al. [1]. Even though the ill-posedness causes difficulty for numerical computation, such problems usually appear in many engineering applications related to propagating waves in different environments, such as acoustic, hydrodynamic and electromagnetic waves [7,8]. Besides, most of those problems are usually solved in 3D domains with inhomogeneous sources, and particularly, source functions depend on the unknown

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function. Hence a necessity of investigating Cauchy problems for nonlinear elliptic equations motivates this study.

Let $a, b, c > 0$ and $\Omega = (-a, a) \times (-b, b)$ be a rectangle in $\mathbb{R}^2$ with the boundary $\partial\Omega$. We seek a function u that satisfies:

$$\Delta u = f(u, x, y, z), \qquad (x, y, z) \in \bar{\Omega} \times [0, c], \tag{1}$$

$$u(x, y, z) = 0, \qquad (x, y, z) \in \partial\Omega \times [0, c], \tag{2}$$

$$\|g^{\varepsilon} - u(\cdot, \cdot, 0)\| + \|h^{\varepsilon} - \partial_z u(\cdot, \cdot, 0)\| \leqslant \varepsilon, \tag{3}$$

where Δ is the three dimensional Laplacian, ∂_z is the z-partial derivative, f is a given function depending on the unknown u, both g^{ε} and h^{ε} are given functions in the Lebesgue space $L^2(\Omega)$ that is endowed with the natural inner product $\langle \cdot, \cdot \rangle$ and with the norm $\|\cdot\|$, the positive number ε is the data error of $(g^{\varepsilon}, h^{\varepsilon})$ from the exact Cauchy data

$$(g, h) = (u, \partial_z u)|_{z=0}, \qquad (x, y) \in \bar{\Omega}. \tag{4}$$

Throughout this paper, we shall use the following notations. The space $H^m(\Omega)$ is the Sobolev space involving all functions that are in $L^2(\Omega)$ as well as their s-th order derivatives for all $s \leqslant m$, $H_0^1(\Omega)$ contains all the functions of $H^1(\Omega)$ such that their traces vanish on $\partial\Omega$. We shall adopt the notation $(C([0, c]; L^2(\Omega)), |||\cdot|||)$ for the Banach space of continuous functions mapping $[0, c]$ to $L^2(\Omega)$, wherein $|||\cdot|||$ stands for the supremum norm. Let λ_{mn}^2 and ψ_{mn} denote the eigenvalues and corresponding eigenfunctions of $A := -\Delta$ defined on its domain $\mathrm{D}(A) \subset H_0^1(\Omega)$, which are

$$\lambda_{mn} = \frac{\pi}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \quad \psi_{mn}(x, y) = \sin\left(\frac{m\pi(x + a)}{2a}\right)\sin\left(\frac{n\pi(y + b)}{2b}\right), \tag{5}$$

for any $(m, n) \in \mathbb{N}^2$. Here after, we shall denote the Fourier coefficients of the functions $v = v(x, y)$, $w = w(x, y, z)$ and $f = f(w, x, y, z)$ by $\hat{v}_{mn} = \kappa \langle v, \psi_{mn} \rangle$, $\hat{w}_{mn}(z) = \kappa \langle w(\cdot, \cdot, z), \psi_{mn} \rangle$ and $\hat{f}_{mn}(w, z) = \kappa \langle f(w(\cdot, \cdot, z), \cdot, \cdot, z), \psi_{mn} \rangle$, respectively, where $\kappa = \|\psi_{mn}\|^{-2} = 1/(ab)$. Similarly, $\partial_z \hat{w}_{mn}(z)$ will stand for $\kappa \langle \partial_z w(\cdot, \cdot, z), \psi_{mn} \rangle$.

Using the method of separation of variables, the exact solution of the problem (1)–(4) satisfies

$$u(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\widehat{G}_{mn}(g, h, z) + \widehat{J}_{mn}(u, z)\right)\psi_{mn}(x, y), \tag{6}$$

where

$$\widehat{G}_{mn}(g, h, z) = \frac{e^{z\lambda_{mn}}}{2}\left(\widehat{g}_{mn} + \frac{\widehat{h}_{mn}}{\lambda_{mn}}\right) + \frac{e^{-z\lambda_{mn}}}{2}\left(\widehat{g}_{mn} - \frac{\widehat{h}_{mn}}{\lambda_{mn}}\right), \tag{7}$$

$$\widehat{J}_{mn}(u, z) = \frac{1}{2\lambda_{mn}}\int_0^z \left(e^{(z-s)\lambda_{mn}} - e^{(s-z)\lambda_{mn}}\right)\widehat{f}_{mn}(u, s)\,\mathrm{d}s. \tag{8}$$

It can be seen that $\hat{G}_{mn}(g, h, z)$ and $\hat{J}_{mn}(u, z)$ in Eqs. (7) and (8) blow up quickly as λ_{mm} tends to infinity due to the fast increasing of $\exp(z\lambda_{mn})$. Therefore, a numerical calculation of Eqs. (6)–(8) in practice is impossible, even when the exact Fourier coefficients $(\hat{g}_{mn}, \hat{h}_{mn}, \hat{f}_{mn})$ may tend to zero rapidly. It is well-known that the Cauchy problem for elliptic equations is severely ill-posed in the sense of Hadamard, i.e., a small perturbation in the given Cauchy data may cause a very large error in the output solution