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Analogues of the Ramanujan–Mordell theorem

Shaun Cooper [a], Ben Kane [b,*,1], Dongxi Ye [c]

[a] Institute of Natural and Mathematical Sciences, Massey University-Albany, Private Bag 102904, North Shore Mail Centre, Auckland, New Zealand
[b] Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong
[c] Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI, 53706, USA

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A B S T R A C T

The Ramanujan–Mordell Theorem for sums of an even number of squares is extended to other quadratic forms and quadratic polynomials.

1. Introduction

One of the classical problems in number theory is to determine exact formulas for the number of representations of a positive integer n as a sum of $2k$ squares, which we denote by $r(2k;n)$. If we set

$$z = z(\tau) := \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+n^2} \tag{1.1}$$

where, here and throughout the remainder of this work, τ is a complex number with positive imaginary part and $q = e^{2\pi i\tau}$, then (considering z as a power series in q)

$$\sum_{n=0}^{\infty} r(2k;n)q^n = z^k.$$

* Corresponding author.
E-mail addresses: s.cooper@massey.ac.nz (S. Cooper), bkane@maths.hku.hk (B. Kane), lawrencefrommath@gmail.com (D. Ye).

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The function z^k is a modular form and it is well known that

$$z^k(\tau) = E_k^*(\tau) + C_k(\tau),$$

where $E_k^*(\tau)$ is an Eisenstein series and $C_k(\tau)$ is a cusp form. In his remarkable work, Ramanujan [24, Eqs. (145)–(147)] stated without proof explicit formulas for $E_k^*(\tau)$ and $C_k(\tau)$, and hence deduced the value of the coefficients $r(2k; n)$. Ramanujan's result was first proved by Mordell [20]. To state it we need Dedekind's eta function, which is defined by

$$\eta(\tau) :=q^{1/24} \prod_{j=1}^{\infty}(1-q^n).$$

Here and throughout, we write η_m for $\eta(m\tau)$ for any positive integer m.

Theorem 1.1 *(Ramanujan–Mordell). Suppose k is a positive integer. Let z be defined by (1.1). Then*

$$z^k = F_k(\tau) + z^k \sum_{1\le j\le \frac{(k-1)}{4}} c_{j,k} x^j \tag{1.2}$$

where $c_{j,k}$ are numerical constants that depend on j and k,

$$x = x(\tau) := \frac{\eta_1^{24}\eta_4^{24}}{\eta_2^{48}},$$

and $F_k(\tau)$ is an Eisenstein series defined by:

$$F_1(\tau) := 1 + 4\sum_{j=1}^{\infty} \frac{q^j}{1+q^{2j}},$$

and for $k \ge 1$,

$$F_{2k}(\tau) := 1 - \frac{4k(-1)^k}{(2^{2k}-1)\mathcal{B}_{2k}} \sum_{j=1}^{\infty} \frac{j^{2k-1}q^j}{1-(-1)^{k+j}q^j},$$

and

$$F_{2k+1}(\tau) := 1 + \frac{4(-1)^k}{\mathcal{E}_{2k}} \sum_{j=1}^{\infty} \left(\frac{(2j)^{2k}q^j}{1+q^{2j}} - \frac{(-1)^{k+j}(2j-1)^{2k}q^{2j-1}}{1-q^{2j-1}} \right).$$

Here $\mathcal{B}_k$ and $\mathcal{E}_k$ are the Bernoulli numbers and Euler numbers, respectively, defined by

$$\frac{u}{e^u-1} = \sum_{k=0}^{\infty} \frac{\mathcal{B}_k}{k!}u^k \quad \textit{and} \quad \frac{1}{\cosh u} = \sum_{k=0}^{\infty} \frac{\mathcal{E}_k}{k!}u^k.$$

The reader is referred to [9, p. 2] for a brief account of the history of the study of Theorem 1.1 for various k.

The goal of this work is to prove the analogues of the Ramanujan–Mordell Theorem for which the quadratic form m^2+n^2 in (1.1) is replaced with the quadratic form m^2+pn^2, or by the quadratic polynomial