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Existence of solutions for semilinear problems with prescribed number of zeros on exterior domains

Joseph A. Iaia

P.O. Box 311430, Department of Mathematics, University of North Texas, Denton, TX 76203-1430, United States

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ABSTRACT

In this paper we prove the existence of an infinite number of radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius R centered at the origin in $\mathbb{R}^N$ such that $\lim_{r\to\infty} u(r) = 0$ with prescribed number of zeros where $f : \mathbb{R} \to \mathbb{R}$ is odd and there exists a $\beta > 0$ with $f < 0$ on $(0, \beta)$, $f > 0$ on (β, ∞) with f superlinear for large u, and $K(r) \sim r^{-\alpha}$ with $0 < \alpha < N$.

1. Introduction

In this paper we study radial solutions of:

$$\Delta u + K(|x|)f(u) = 0 \text{ for } R < |x| < \infty, \tag{1}$$

$$u(x) = 0 \text{ when } |x| = R, \lim_{|x|\to\infty} u(x) = 0, \tag{2}$$

where $u : \mathbb{R}^N \to \mathbb{R}$ with $N \geq 2$, $R > 0$, f is odd and locally Lipschitz with:

$$f'(0) < 0, \exists\, \beta > 0 \text{ s.t. } f(u) < 0 \text{ on } (0, \beta),\ f(u) > 0 \text{ on } (\beta, \infty). \tag{H1}$$

In addition, we assume:

$$f(u) = |u|^{p-1}u + g(u) \text{ where } p > 1 \text{ and } \lim_{u\to\infty} \frac{|g(u)|}{|u|^p} = 0. \tag{H2}$$

E-mail address: iaia@unt.edu.

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Denoting $F(u) \equiv \int_0^u f(t)\,dt$ we assume:

$$\exists\, \gamma \text{ with } 0 < \beta < \gamma \text{ s.t. } F < 0 \text{ on } (0, \gamma) \text{ and } F > 0 \text{ on } (\gamma, \infty). \tag{H3}$$

Further we also assume K and K' are continuous on $[R, \infty)$ and:

$$K(r) > 0, \exists\, \alpha \in (0, 2(N-1)) \text{ s.t. } \lim_{r\to\infty} \frac{rK'}{K} = -\alpha, \text{ and } \exists \text{ positive } d_1, d_2 \tag{H4}$$

$$\text{s.t. } 2(N-1) + \frac{rK'}{K} > 0,\ d_1 r^{-\alpha} \le K(r) \le d_2 r^{-\alpha} \text{ for } r \ge R. \tag{H5}$$

Main Theorem. *Assuming (H1)–(H5), $N \ge 2$, and $0 < \alpha < N$ then for each nonnegative integer n there exists a radial solution, u_n, of (1)–(2) such that u_n has exactly n zeros on (R, ∞).*

The radial solutions of (1) on $\mathbb{R}^N$ and $K(r) \equiv 1$ have been well-studied. These include [1,2,7,9,11]. Recently there has been an interest in studying these problems on $\mathbb{R}^N \backslash B_R(0)$. These include [4,5,8,10]. Here we use a scaling argument as in [9] to prove existence of solutions.

2. Preliminaries

Since we are interested in radial solutions of (1)–(2), we denote $r = |x|$ and write $u(x) = u(|x|)$ where u satisfies:

$$u'' + \frac{N-1}{r} u' + K(r) f(u) = 0 \text{ for } R < r < \infty, \tag{3}$$

$$u(R) = 0, u'(R) = b > 0. \tag{4}$$

We will occasionally write $u(r, b)$ to emphasize the dependence of the solution on b. By the standard existence-uniqueness theorem [3] there is a unique solution of (3)–(4) on $[R, R+\epsilon)$ for some $\epsilon > 0$.

We next consider:

$$E(r) = \frac{1}{2} \frac{u'^2}{K(r)} + F(u). \tag{5}$$

It is straightforward using (3) and (H5) to show:

$$E'(r) = -\frac{u'^2}{2rK}[2(N-1) + \frac{rK'}{K}] \le 0. \tag{6}$$

Thus E is non-increasing. Therefore:

$$\frac{1}{2} \frac{u'^2}{K(r)} + F(u) = E(r) \le E(R) = \frac{1}{2} \frac{b^2}{K(R)} \text{ for } r \ge R. \tag{7}$$

Since F is bounded from below by (H3), it follows from (7) that u and u' are uniformly bounded wherever they are defined from which it follows that the solution of (3)–(4) is defined on $[R, \infty)$.

Lemma 2.1. *Let u be the solution of (3)–(4). Assume (H1)–(H5) and $N \ge 2$. If $b > 0$ and b is sufficiently small then $u(r) > 0$ for all $r > R$.*