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Higher integrability of Green's operator and homotopy operator

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ABSTRACT

We prove the higher integrability and higher order imbedding theorems of the composition of homotopy operator and Green's operator. We also study analogues of the homotopy operator and Green's operator, respectively.

1. Introduction

The purpose of this paper is to prove the higher integrability and higher order imbedding theorems of the composition of the homotopy operator T and Green's operator G applied to differential forms. We all know that the integrability and differentiability are quite universal concepts which are well studied and widely used in many fields of mathematics and physics, such as analysis, dynamical systems, quantum physics and partial differential equations. The integrability of various functions and operators is an essential and core topic in different areas of mathematics, including analysis and partial differential equations. The recent development of the theory of integrability influenced different branches of mathematics and mathematical physics, see [11–13,21,24] for example. It is very often for us to deal with the integrability of the operators and to estimate the upper bounds for the norms of these operators. For instance, while studying the L^p-theory of differential forms and related operators and investigating the qualitative and quantitative properties of the solutions of partial differential equations as well as operator equations in certain situations, we have to explore the L^p integrability of functions, differential forms and the related operators and their compositions, such as in the cases of the Hodge decomposition and the well-known Poisson's equation. The integrability of the composite operators is more complicated than that of the single operator. It is well known that any

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differential form u can be decomposed as $u = dT(u) + Td(u)$, where d is the exterior derivative and T is the homotopy operator applied to differential forms. Hence, we have to deal with the composite operator $T \circ G$ when we consider the decomposition of $G(u)$, where G is Green's operator applied to differential form u. In previous related work, the investigation has been mainly focused on estimating the L^p-norms of the operators applied to the differential form u in terms of the L^p-norms of u, see [2,6,10,16,17]. For example, one can estimate $\|G(u)\|_p$ in terms of the norm $\|u\|_p$. However, if $s > p$, one could not estimate $\|G(u)\|_s$ in terms of the norm $\|u\|_p$ of u using existing results. That is, from the existing literature, we are not sure whether $G(u)$ has a higher integrability than u does. In this paper, we concentrate our attention on the higher integrability and the higher order imbedding theorems of the composition of the two key operators, the homotopy operator T and Green's operator G applied to differential forms. The significance of the higher integrability and the higher order imbeddings of these operators might be far reaching since these new results will develop broad and rigorous conceptual understanding of the L^p-theory of differential forms and related operators. They can be used to derive the regularity theory for solutions of the partial differential equations and to control oscillatory behavior of the solutions of the harmonic equations.

Differential forms have become invaluable tools for many fields of sciences and engineering, such as theoretical physics and general relativity. They can be used to describe various systems of partial differential equations and to express different geometric structures on manifolds, see [1–4,8–10]. For instance, some kinds of differential forms are often utilized in studying deformations of elastic bodies, the related extrema for variational integrals, and certain geometric invariance. The L^p-theory of operators on differential forms has been very well developed and various L^p-norm inequalities and estimates have been established during the recent years, see [6,10,16,17]. Our main results, the global higher integrability of the composite operator $G \circ T$, the homotopy operator T and Green's operator G are presented in Section 3. We prove that $GT(u)$, $T(u)$ and $G(u)$ have higher integrability than u does.

Differential forms are extensions of differentiable functions in $\mathbb{R}^n$. For example, a function $u(x_1, x_2, \cdots, x_n)$ is called a 0-form. A differential k-form $u(x)$ is generated by $\{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\}$, $k = 1, 2, \cdots, n$, that is, $u(x) = \sum_I \omega_I(x)dx_I = \sum \omega_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, i_2, \cdots, i_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$. Much work has been done for differential forms satisfying some versions of harmonic equations, see [22,23,25,26] for example. We keep using the same notations appearing in [1]. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a domain with $|\Omega| < \infty$, B and σB be the balls with the same center and $\text{diam}(\sigma B) = \sigma\text{diam}(B)$. We do not distinguish the balls from cubes in this paper. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and $L^p(\Omega, \wedge^l)$ be the set of all l-forms $u(x) = \sum_I u_I(x)dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \cdots, n$. We denote the exterior derivative by d and the Hodge star operator by $\star$. The Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d \star$, $l = 1, 2, \cdots, n$. For $u \in D'(\Omega, \wedge^l)$ the vector-valued differential form $\nabla u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$ consists of differential forms $\frac{\partial u}{\partial x_i} \in D'(\Omega, \wedge^l)$, where the partial differentiation is applied to the coefficients of u. Let $|E|$ be the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. For a function u, the average of u over B is defined by $u_B = \frac{1}{|B|}\int_B u dx$. All integrals involved in this paper are the Lebesgue integrals. We use $C^\infty(\Omega, \wedge^l)$ to denote the space of smooth l-forms on Ω and the Green's operator G be defined on $C^\infty(\Omega, \wedge^l)$ by assigning $G(u)$ to be a solution of the Poisson's equation $\Delta G(u) = u - H(u)$, where H is the harmonic projection operator, see [1] and [19] for more results about Green's operator. For any subset $E \subset \mathbb{R}^n$ and $p > 0$, we use $W^{1,p}(E, \wedge^l)$ to denote the Sobolev space of l-forms which equals $L^p(E, \wedge^l) \cap L^p_1(E, \wedge^l)$ with norm

$$\|u\|_{W^{1,p}(E)} = \|u\|_{W^{1,p}(E,\wedge^l)} = diam(E)^{-1}\|u\|_{p,E} + \|\nabla u\|_{p,E}. \tag{1.1}$$

A homotopy operator $T : C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ is defined by averaging K_y over all points $y \in \Omega$: $Tu = \int_\Omega \phi(y)K_y u dy$, where $\phi \in C_0^\infty(\Omega)$ is normalized so that $\int \phi(y)dy = 1$, and the linear operator K_y : $C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ is defined by $(K_y u)(x; \xi_1, \cdots, \xi_{l-1}) = \int_0^1 t^{l-1}u(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1})dt$.