On generalized m-quasi-Einstein manifolds with constant Ricci curvatures ☆

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A B S T R A C T

In this paper, we study generalized m-quasi-Einstein manifold with constant Ricci curvatures. In particular, as the main result we show that for $m \neq 1$ a proper generalized m-quasi-Einstein manifold with constant Ricci curvatures is Einstein. Consequently for $m \neq 1$ every homogeneous proper generalized m-quasi-Einstein manifold must be Einstein.

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1. Introduction

The study of Einstein manifolds and their various generalizations is always an attractive topic in modern Riemannian geometry. In particular, there has been increasing interest on so-called quasi-Einstein manifolds. Recall that a triple (M^n, g, f), i.e., Riemannian manifold (M^n, g) with a potential function f on M, is called m-quasi-Einstein, if its associated m-Bakry–Emery Ricci tensor $\text{Ric}_m^f := \text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df$ is a constant multiple of the metric g (cf. [6] and the references therein). The study of m-quasi-Einstein manifolds becomes interesting partially due to that, according to [16], an n-dimensional m-quasi-Einstein manifold is exactly the manifold which is the base of an $(n+m)$-dimensional Einstein warped product with m-dimensional Einstein fiber. To extend the notion of m-quasi-Einstein, Catino [7] introduced the concept of generalized quasi-Einstein manifold, and as its particular case, Barros and Ribeiro [2] further proposed to consider the following notion of generalized m-quasi-Einstein manifold:

Definition 1.1 *([2]).* Let (M^n, g) be an n-dimensional Riemannian manifold and m a positive integer. If there exist two smooth functions f and λ on M such that the Ricci tensor Ric of (M^n, g) satisfies the relation

$$\text{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{1.1}$$

where ∇^2 and $\otimes$ denote the Hessian and the tensorial product, respectively, then g is called a generalized m-quasi-Einstein metric and $(M^n, g, \nabla f, \lambda)$ a generalized m-quasi-Einstein manifold. If, in particular, the function λ in (1.1) is non-constant on M, the generalized m-quasi-Einstein manifold is called proper.

In recent years, generalized m-quasi-Einstein manifolds have been extensively studied by many mathematicians, see e.g. [1,2,7,8,12,14,15,18,21], among many others. It is noteworthy to point out that if $m = \infty$ and λ is a constant, (1.1) becomes the equation satisfied by the gradient Ricci solitons which play a crucial role in Hamilton's Ricci flow. As an important aspect in the treatment of the Ricci flow, subject of gradient Ricci solitons has been treated by many authors, we refer to Cao [5] for a complete survey up to that time. If $m = \infty$, (1.1) becomes the Ricci almost soliton equation (cf. [20]). In addition, if particularly in (1.1), $\lambda = \rho S + \mu$, where S denotes the scalar curvature of (M^n, g) and ρ, μ are two real constants, then (M^n, g) becomes an (m, ρ)-quasi-Einstein manifold which is closely related to the so-called ρ-Einstein flows (cf. [13]).

Considering the positive function $u = e^{-\frac{f}{m}}$ on M, then (1.1) can be rewritten as

$$\text{Ric} - \frac{m}{u} \nabla^2 u = \lambda g. \tag{1.2}$$

So the generalized m-quasi-Einstein manifold $(M^n, g, \nabla f, \lambda)$ that satisfies (1.1) is usually also denoted by (M^n, g, u, λ), where $u = e^{-\frac{f}{m}}$. Moreover, (M^n, g, u, λ) will be called *trivial* if the potential function u is constant. Otherwise, it will be called *nontrivial*. It is easy to see that an n-dimensional ($n \geq 3$) proper generalized m-quasi-Einstein manifold must be nontrivial. Obviously, the triviality of (M^n, g, u, λ) implies that (M^n, g) is Einstein, but generally the converse is not true. To see this fact more clearly, we would recall that in [2] it was shown that, as trivially Einstein manifolds, each of the three kinds of space forms can also possess proper generalized m-quasi-Einstein structure. Also in [2], Barros and Ribeiro studied generalized m-quasi-Einstein manifolds and as the main result they showed that an Einstein nontrivial generalized m-quasi-Einstein manifold must be isometric to one of the space forms with the appropriate functions f and λ. In [12], we generalize the result of [2] by considering generalized m-quasi-Einstein metrics with parallel Ricci tensor. As the result we obtain the following classification theorem.

Theorem 1.1 *([12]). Let (M^n, g, u, λ) be a complete n-dimensional ($n \geq 3$) nontrivial generalized m-quasi-Einstein manifold which possesses parallel Ricci tensor. Then (M^n, g) is isometric to one of the following manifolds:*

(1) *a space form,*
(2) $\mathbb{D}_c^n$,
(3) $\mathbb{R} \times N^{n-1}(b)$,
(4) $\mathbb{H}^p(a) \times N^{n-p}(b)$, $b = \frac{m+p-1}{p-1} a$,
(5) $\mathbb{D}_c^p \times N^{n-p}(b)$, $b = (1 - m - p)c^2$,

where a, b are negative constants, $N^k(b)$ denotes a k-dimensional Einstein manifold with scalar curvature kb, here we recall that traditionally one also calls b the Einstein constant of $N^k(b)$; $\mathbb{H}^p(a)$ denotes the p-dimensional hyperbolic space with Einstein constant a; $\mathbb{D}_c^k$ denotes a k-dimensional Einstein warped product $\mathbb{R} \times_{c^{-1}e^{cr}} F^{k-1}$, i.e. $\mathbb{R} \times F^{k-1}$ endowed with the metric $dr^2 + (c^{-1}e^{cr})^2 g_F$, c is a positive constant, F^{k-1} with metric g_F is a Ricci flat manifold.

In this paper, as the continuation towards the problem being settled by Theorem 1.1, we will focus on a more general case of nontrivial generalized m-quasi-Einstein metric with constant Ricci curvatures, i.e.,