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Norm-inflation results for the BBM equation

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ABSTRACT

Considered here is the periodic initial-value problem for the regularized long-wave (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

Adding to previous work in the literature, it is shown here that for any $s < 0$, there is smooth initial data that is small in the L_2-based Sobolev spaces H^s, but the solution emanating from it becomes arbitrarily large in arbitrarily small time. This so called *norm inflation* result has as a consequence the previously determined conclusion that this problem is ill-posed in these negative-norm spaces.

1. Introduction

This note derives from the paper [6] where it was shown that the initial-value problem

$$\begin{aligned} &u_t + u_x + uu_x - u_{xxt} = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

for the regularized long-wave or BBM equation is globally well posed in the L_2-based Sobolev spaces $H^r(\mathbb{R})$ provided $r \geq 0$. In the same paper, it was shown that the map that takes initial data to solutions cannot be locally C^2 if $r < 0$. This latter result suggests, but does not prove, that the problem (1.1) is not well posed in H^r for negative values of r. Later, Panthee [14] showed that this solution map, were it to exist on all of $H^r(\mathbb{R})$, could not even be continuous, thus proving that the problem is ill posed in the L_2-based Sobolev spaces with negative index. Indeed, he showed that there is a sequence of smooth initial data $\{\phi_n\}_{n=1}^{\infty}$ such

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that $\phi_n \to 0$ in $H^r(\mathbb{R})$ but the associated solutions, $\{u_n\}_{n=1}^{\infty}$ have the property that $\|u(\cdot,t)\|_{H^r}$ is bounded away from zero for all small values of $t > 0$ and all $n \geq 1$.

The BBM equation itself was initially put forward in [15] and [2] as an approximate description of long-crested, surface water waves. It is an alternative to the classical Korteweg–de Vries equation and has been shown to be equivalent in that, for physically relevant initial data, the solutions of the two models differ by higher order terms on a long time scale (see [5].) It predicts the propagation of surface water waves pretty well in its range of validity [4]. Finally, it is known rigorously to be a good approximation to solutions of the full, inviscid, water-wave problem by combining results in [1,3] and [12] (see also [13]).

It is our purpose here to show that in fact, for $r < 0$, the problem (1.1) is not only not well posed, but features blow-up in the H^r-norm in arbitrarily short time. This will be done in the context of the periodic initial-value problem wherein u_0 is a periodic distribution lying in H^r_{per} for some $r < 0$. Similar results hold for $H^r(\mathbb{R})$, but are not explicated here.

More precisely, it will be shown that, for any given $r < 0$, there is a sequence $\{u_0^n\}_{n=1}^{\infty}$ of smooth initial data such that $u_0^n \to 0$ in H^r_{per} and a sequence $\{T_n\}_{n=1}^{\infty}$ of positive times tending to 0 as $n \to \infty$ such that the corresponding solutions $\{u_n\}_{n=1}^{\infty}$ emanating from this initial data, whose existence is guaranteed by the periodic version [8] of the theory for the initial-value problem, are such that for $n = 1, 2, 3, \cdots$,

$$\|u(\cdot, T_n)\|_{H^s_{per}} \geq n.$$

This insures in particular that the solution map $\mathcal{S}$ that associates solutions to initial data, which exists on L_2, cannot be extended continuously to all of H^s_{per}, thus reproducing Panthee's conclusion. Results of this sort go by the appellation *norm inflation* for obvious reasons. The idea originated in the work of Bourgain and Pavlović [7] for the three-dimensional Navier–Stokes equation. The method of construction there was applied to some other dissipative fluid equations by the second author and her collaborators, see [11,10,9]. It suggests that the method is generic as well as sophisticated.

Notation The notation used throughout is standard. For $r \in \mathbb{R}$, the collection $\dot{H}^r_{per}$ is the homogeneous space of 2π-periodic distributions whose norm

$$\|f\|_r^2 = \sum_{k=1}^{\infty} k^{2r}(|f_k|^2 + |g_k|^2)$$

is finite. Elements in $\dot{H}^r_{per}$ all have mean zero over the period domain $[0, 2\pi]$. Here, the $\{f_k\}$ are the Fourier sine coefficients and the $\{g_k\}$ are the Fourier cosine coefficients of f. Notice that $\dot{H}^0_{per}$ may be viewed simply as the L_2-functions on the period domain $[0, 2\pi]$ with mean zero. If X is any Banach space, the set $C([0,T]; X)$ consists of the continuous functions from the real interval $[0,T]$ into X with its usual norm.

2. Norm inflation

The principal result of our study is the following theorem.

Theorem 2.1. *Let $r < 0$ by given. Then there is a sequence $\{u_0^j\}_{j=1}^{\infty}$ of $\mathcal{C}^{\infty}$, periodic initial data such that*

$$u_0^{(j)} \to 0 \quad \text{as} \quad j \to \infty$$

in $\dot{H}^r_{per}$ and a sequence $\{T_j\}_{j=1}^{\infty}$ of positive times tending to zero as $j \to \infty$ such that if $u_j(x,t)$ is the solution emanating from $u_0^{(j)}$, then