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Baire theorem for ideals of sets ☆

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ABSTRACT

We study ideals $\mathcal{I}$ on $\mathbb{N}$ satisfying the following Baire-type property: *if X is a complete metric space and $\{X_A \colon A \in \mathcal{I}\}$ is a family of nowhere dense subsets of X with $X_A \subset X_B$ whenever $A \subset B$, then $\bigcup_{A\in\mathcal{I}} X_A \neq X$.* We give several characterizations and determine the ideals having this property among certain classes like analytic ideals and P-ideals. We also discuss similar covering properties when considering families of compact and meager subsets of X.

1. Introduction

For a given set X we denote as usual by $\mathcal{P}(X)$ the collection of all subsets of X. We call a set $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ an *ideal* if $\mathbb{N} \notin \mathcal{I}$ and given $A, B \in \mathcal{I}$ we have that $\mathcal{P}(A) \subset \mathcal{I}$ and $A \cup B \in \mathcal{I}$. A set $\beta \subset \mathcal{I}$ is a basis of $\mathcal{I}$ if every $A \in \mathcal{I}$ is contained in some $B \in \beta$. The *character* of $\mathcal{I}$ is the minimal cardinality of a basis of $\mathcal{I}$. Along this paper, every considered ideal $\mathcal{I}$ is supposed to contain the ideal Fin of all finite subsets of $\mathbb{N}$.

As a dual concept, a set $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is a *filter* on $\mathbb{N}$ if $\{\mathbb{N} \setminus A \colon A \in \mathcal{F}\}$ is an ideal on $\mathbb{N}$, and $\beta \subset \mathcal{F}$ is called a basis of $\mathcal{F}$ if every $A \in \mathcal{F}$ contains some $B \in \beta$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a topological space X, then it is said to be *$\mathcal{F}$-convergent* to $a \in X$, usually written $a = \lim_{n,\mathcal{F}} x_n$, if for every neighbourhood V of a we have that $\{n \in \mathbb{N} \colon x_n \in V\}$ belongs to $\mathcal{F}$.

Let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let E be an arbitrary Banach space. A sequence $(e_n)_{n\in\mathbb{N}}$ in E is said to be an *$\mathcal{F}$-basis* of E if for every $x \in E$ there exists a unique sequence of scalars $(a_n)_{n\in\mathbb{N}}$ such that

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$$x = \lim_{n,\mathcal{F}} \sum_{i=1}^{n} a_i e_i.$$

This definition extends the notion of Schauder basis, which corresponds to the case $\mathcal{F}_{cf} := \{A \subset \mathbb{N} \colon \mathbb{N} \setminus A \in \mathrm{Fin}\}$, known as the Fréchet filter. The concept of $\mathcal{F}$-basis was introduced in [5], but previously considered in [2] for the filter of statistical convergence

$$\mathcal{F}_{st} := \left\{ A \subset \mathbb{N} \colon \lim_n \frac{|A \cap \{1, \dots, n\}|}{n} = 1 \right\}.$$

It is clear from the definition of $\mathcal{F}$-basis that the coefficient maps $e_n^*(x) = a_n$ are linear on E. However, and in contrast with Schauder bases, it is not known whether the e_n^*'s are necessarily continuous. A partial result was given by T. Kochanek [8], who showed that if $\mathcal{F}$ has character less than $\mathfrak{p}$ then the answer is positive. Here $\mathfrak{p}$ denotes the *pseudointersection number*, defined as the minimum of the cardinals κ for which the following claim is true: *if $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ with cardinality less than κ and satisfying that $\bigcap \mathcal{A}_0$ is infinite for each finite subset $\mathcal{A}_0 \subset \mathcal{A}$, then there is an infinite set $B \subset \mathbb{N}$ such that $B \setminus A$ is finite for every $A \in \mathcal{A}$.*

If we work with the dual ideal $\mathcal{I}$ associated to $\mathcal{F}$, a review of Kochanek's argument shows that the key step to get the result is that $\mathcal{I}$ has the next property:

($\square$) If X is a complete metric space and $\{X_A \colon A \in \mathcal{I}\}$ is a set of meager subsets of X with $X_A \subset X_B$ whenever $A \subset B$, then $\bigcup \{X_A \colon A \in \mathcal{I}\} \neq X$.

Unfortunately not every ideal has this property. If the character of $\mathcal{I}$ is less than $\mathfrak{p}$, then it has property ($\square$), since $\bigcup \{X_A : A \in \mathcal{I}\}$ is equal to $\bigcup \{X_B : B \in \beta\}$ which is the union of less than $\mathfrak{p}$ meager subsets, and this is again a meager subset of X by [4, Corollary 22C]. In section 2, we show that the converse is also true under the set-theoretical assumption $\mathfrak{p} = \mathfrak{c}$.

The aim of this paper is to study what happens if we replace the condition "meager" by "nowhere dense" in ($\square$). Ideals satisfying this last property will be called *Baire ideals*. In section 3 we prove several characterizations of this type of ideals. We also show that in order to demonstrate that an ideal $\mathcal{I}$ is a Baire ideal, we just have to check property ($\square$) (with nowhere dense subsets instead of meager ones) for the metrizable space $X = D^{\mathbb{N}}$, D being the discrete space of cardinality equal to $\mathfrak{c}$.

The fourth section is devoted to determine which are the Baire ideals in the classes of analytic ideals and P-ideals. Here we work in ZFC without any other set-theoretical assumptions. We show that in both cases the only Baire ideals are the countably generated ideals. In contrast with this, we construct in section 5 a model of ZFC in which we can find an uncountably generated P-ideal $\mathcal{I}$ satisfying property ($\square$) for the particular case $X = 2^{\mathbb{N}}$. We also study the case of ideals generated by an almost disjoint family of subsets of $\mathbb{N}$.

In the last section, we show that if in ($\square$) one considers compact subsets instead of meager ones, then there are uncountably generated F_σ ideals satisfying that property for $X = \mathbb{N}^{\mathbb{N}}$.

Our notation and terminology is standard and it is either explained when needed or can be found in [6] and [7].

2. Baire ideals

As we announced in the introduction, if we assume that $\mathfrak{p} = \mathfrak{c}$ (for instance, under Martin's Axiom), property ($\square$) depends exclusively on the character of the ideal $\mathcal{I}$, as the following proposition shows.

Proposition 2.1. *If $\mathcal{I}$ has a basis of cardinality less than $\mathfrak{p}$ then it has property ($\square$). If $\mathfrak{p} = \mathfrak{c}$, then the converse is true.*