J. Math. Anal. Appl. 445 (2017) 1258–1266

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Numerical radius attaining compact linear operators ☆

Ángela Capel [a], Miguel Martín [b,*], Javier Merí [b]

[a] *Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, C/ Nicolás Cabrera 13-15, 28049 Madrid, Spain*
[b] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*

ARTICLE INFO

Article history:
Received 12 November 2015
Available online 2 March 2016
Submitted by J.A. Ball

Dedicated to Richard Aron on the occasion of his retirement from Kent State University

Keywords:
Banach space
Numerical radius attaining
Norm attaining
Compact linear operator

ABSTRACT

We show that there are compact linear operators on Banach spaces which cannot be approximated by numerical radius attaining operators.
© 2016 Elsevier Inc. All rights reserved.

1. Introduction

Given a real or complex Banach space X, we write S_X and B_X to denote its unit sphere and its unit ball, respectively, and X^* for the topological dual space of X. If Y is another Banach space, $L(X,Y)$ denotes the space of all bounded linear operators from X to Y and we just write $L(X)$ for $L(X,X)$. The space of all compact linear operators on X will be denoted by $K(X)$. We consider the set

$$\Pi(X) = \big\{(x,x^*) \in X \times X^* \,:\, x \in S_X,\, x^* \in S_{X^*},\, x^*(x) = 1\big\}.$$

The *numerical range* of $T \in L(X)$ is the subset of the base field given by

$$V(T) = \big\{x^*(Tx) \,:\, (x,x^*) \in \Pi(X)\big\}.$$

☆ First author partially supported by a La Caixa-Severo Ochoa grant (ICMAT Severo Ochoa project SEV-2011-0087, MINECO). Second and third authors partially supported by Spanish MINECO and FEDER project no. MTM2012-31755 and by Junta de Andalucía and FEDER grant FQM-185.
* Corresponding author.
E-mail addresses: angela.capel@icmat.es (Á. Capel), mmartins@ugr.es (M. Martín), jmeri@ugr.es (J. Merí).

http://dx.doi.org/10.1016/j.jmaa.2016.02.074
0022-247X/© 2016 Elsevier Inc. All rights reserved.

A complete survey on numerical ranges and their relations to spectral theory of operators can be found in the books by F. Bonsall and J. Duncan [8,9], where we refer the reader for general information and background. The recent development of this topic can be found in sections 2.1 and 2.9 of [10] and references therein.

The *numerical radius* of $T \in L(X)$ is given by

$$v(T) = \sup\{|\lambda| \,:\, \lambda \in V(T)\}.$$

It is clear that v is a continuous seminorm which satisfies $v(T) \leqslant \|T\|$ for every $T \in L(X)$. It is said that T *attains its numerical radius* when the supremum defining $v(T)$ is actually a maximum. We will denote by $\mathrm{NRA}(X)$ the set of numerical radius attaining operators on X. One clearly has that $\mathrm{NRA}(X) = L(X)$ if X is finite-dimensional. Even in a separable Hilbert space it is not difficult to find diagonal operators which do not attain their numerical radii. Our paper deals with the study of the density of $\mathrm{NRA}(X)$. This study was started in the PhD dissertation of B. Sims of 1972 (see [7]), parallel to the study of norm attaining operators initiated by J. Lindenstrauss in 1963 [18] (recall that a bounded linear operator T between two Banach spaces X and Y is said to *attain its norm* if there is $x \in S_X$ such that $\|Tx\| = \|T\|$). Among the positive results on this topic, we would like to mention that the set of numerical radius attaining operators is dense for Banach spaces with the Radon–Nikodym property (M. Acosta and R. Payá [5]) and for $L_1(\mu)$ spaces (M. Acosta [1]) and $C(K)$ spaces (C. Cardassi [11]). On the other hand, the first example of Banach space for which the set of numerical radius attaining operators is not dense was given by R. Payá in 1992 [26]. Another counterexample was discovered shortly later by M. Acosta, F. Aguirre and R. Payá [4]. The proofs in these two papers are tricky and non-trivial, and, in both examples, the operators shown that cannot be approximated by numerical radius attaining operators are not compact.

Our aim in this paper is to show that there are **compact** linear operators which cannot be approximated by numerical radius attaining operators. The analogous problem about compact operators which cannot be approximated by norm attaining operators has been recently solved by the second author of this manuscript [21]. Actually, the proofs here mix some ideas from that paper with some ideas from the already mentioned counterexamples for the density of numerical radius attaining operators [4,26].

Let us comment that the counterexamples in [4,26] follow similar lines and borrow some ideas from the seminal paper by Lindenstrauss [18]: they are of the form $Y \oplus_\infty Z$, where Y^* is smooth enough, Z fails to have extreme points in its unit ball in a strong way, and there are operators from Z into Y that cannot be approximated by numerical radius attaining operators (norm attaining operators in the counterexample of Lindenstrauss). Moreover, by construction, the operators shown there that cannot be approximated by numerical radius attaining operators are not compact, as they are constructed using operators from Z into Y which are not compact. Actually, the existence of non-compact operators from Z to Y is one of the building blocks of their proofs. Here this fact will be replaced by the use of Banach spaces without the approximation property.

Taking advantage of some recent ideas, it is now easy (up to a non-trivial old result by W. Schachermayer) to present new examples of (non-compact) operators which cannot be approximated by numerical radius attaining operators. However, as we will show at the end of the paper, these new examples do not work for compact operators (see Example 3.4).

Example 1.1. *There are bounded linear operators on the real spaces $X = C[0,1] \oplus_1 L_1[0,1]$ and $Y = C[0,1] \oplus_\infty L_1[0,1]$ which cannot be approximated by numerical radius attaining operators.*

Indeed, suppose for the sake of contradiction that $\mathrm{NRA}(X)$ is dense in $L(X)$. As $v(T) = \|T\|$ for every $T \in L(X)$ ([13, Theorem 2.2] and [23, Proposition 1]), it follows that norm-attaining operators from X into X are dense in $L(X)$. Then, we may use [6, Proposition 2.9] and [27, Lemma 2] to get that norm attaining operators from $L_1[0,1]$ into $C[0,1]$ are dense in $L(L_1[0,1], C[0,1])$, but this is not the case as shown by W. Schachermayer [28]. The proof for Y is absolutely analogous.

Download English Version:

https://daneshyari.com/en/article/4613984

Download Persian Version:

https://daneshyari.com/article/4613984

Daneshyari.com