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Norming points and critical points

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A R T I C L E I N F O

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This paper is dedicated to Professor Richard M. Aron

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A B S T R A C T

Using a diffeomorphism between the unit sphere and a closed hyperplane of an infinite dimensional Banach space, we introduce the differentiation of a function defined on the unit sphere, and show that a continuous linear functional attains its norm if and only if it has a critical point on the unit sphere. Furthermore, we provide a strong version of the Bishop–Phelps–Bollobás theorem for a Lipschitz smooth Banach space.

1. Main results

Let X be a Banach space and S_X be its unit sphere. A continuous linear functional $f \in X^*$ is said to attain its norm if there exists $x_0 \in S_X$ such that $|f(x_0)| = ||f||$, i.e. $|f|$ has a maximum on S_X. The point x_0 is called a "norming point" of f. The first cornerstone in studying norm-attaining linear functionals is James' characterization [8] of a reflexive Banach space, which says that every continuous linear functional on X attains its norm if and only if X is reflexive. After the celebrated Bishop–Phelps theorem [3], "for a Banach space X, the set of all norm-attaining linear functionals is dense in X^*" appeared in 1961, a lot of attention has been paid to the study of this property for linear operators between Banach spaces.

In this short paper we want to show that a norming point $x_0 \in S_X$ of f is a critical point of f, that is, $f'(x_0) = 0$. However, from the concept of the Frechét differentiation of f we have $f'(x_0) = f$. Hence we introduce a concept of the differentiation of a function f defined on S_X, which is compatible with the differentiation on a manifold.

We now assume that X is a Banach space with a C^p smooth norm $(1 \leq p \leq \infty)$. For every $z \in S_X$, we denote by H_{-z} the hyperplane tangent to S_X at $-z$. Let π_z be the stereographic projection from $S_X \setminus \{z\}$

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onto H_{-z}. The desired manifold structure on S_X is defined by the family $\{\pi_z \mid z \in S_X\}$. It is easily checked that S_X is a C^p submanifold, modelled on a codimension one, closed linear subspace X_0 of X [10, II, §2, Example]. Let Ψ be a diffeomorphism from a closed hyperplane H of a Banach space X onto S_X. For $f : S_X \to \mathbb{R}$ we define $Df_\Psi : S_X \to H^*$ by $Df_\Psi(u)(h) := D(f \circ \Psi)(\Psi^{-1}(u))(h)$ for $u \in S_X$ and $h \in H$, where D is the Frechét differentiation. We say that f has a "critical point" at $u \in S_X$ if $Df_\Psi(u) = 0$ for some diffeomorphism Ψ from a closed hyperplane H of a Banach space X onto S_X.

For its well-definedness, it is easy to check that given diffeomorphisms $\Psi_1 : H_1 \to S_X$ and $\Psi_2 : H_2 \to S_X$, $Df_{\Psi_1}(u) = 0$ if and only if $Df_{\Psi_2}(u) = 0$. Indeed,

$$\begin{aligned} D(f \circ \Psi_1)(\Psi_1^{-1}(u)) &= D(f \circ \Psi_2 \circ \Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)) \\ &= D(f \circ \Psi_2)(\Psi_2^{-1}(u)) \circ D(\Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)). \end{aligned}$$

Since $\Psi_2^{-1} \circ \Psi_1$ is a diffeomorphism, $D(\Psi_2^{-1} \circ \Psi_1)(\Psi_1^{-1}(u)) \in \mathcal{L}(H_1, H_2)$ is an isomorphism.

In 1966, C. Bessaga [2] proved that every infinite dimensional Hilbert space is C^∞ diffeomorphic to its unit sphere. By improving Bessaga's non-complete technique, H.T. Dobrowolski [5] proved in 1979 that every infinite dimensional Banach space X which is linearly injectable into some $c_0(\Gamma)$ is C^∞ diffeomorphic to $X \setminus \{0\}$. More generally, Azagra [1] showed the following result in 1997.

Theorem 1.1. *(See [1, Theorem 1].) Let X be an infinite dimensional Banach space with a C^p smooth norm, where $p \in \mathbb{N} \cup \{\infty\}$. Then for every closed hyperplane H in X, there exists a C^p diffeomorphism between S_X and H.*

From now on, let Ψ denote the C^p diffeomorphism from H onto S_X given in the above theorem.

Theorem 1.2. *Let X be an infinite dimensional Banach space with a C^p smooth norm ($1 \le p \le \infty$). Then $f \in S_{X^*}$ attains its norm at $u \in S_X$ if and only if $f|_{S_X}$ has a critical point at $u \in S_X$.*

Proof. Suppose f attains its norm at $u \in S_X$. For each vector $h \in H$, $\|h\| = 1$ we define $g : \mathbb{R} \to \mathbb{R}$ by $g(\lambda) = f \circ \Psi(\lambda h + \Psi^{-1}(u))$. Then it is clear that g is differentiable on $\mathbb{R}$ and also attains either a maximum or a minimum at $\lambda = 0$. Hence $g'(0) = 0 = D(f \circ \Psi)(\Psi^{-1}(u))(h)$, which implies that $Df_\Psi(u) = 0$.

For the proof of the other implication it is enough to show that if $0 \le f(x_1) < 1$ at $x_1 \in S_X$, then $Df_\Psi(x_1) \neq 0$. Choose $x_2 \in S_X$ with $f(x_2) > f(x_1)$ and define $\gamma : [0, 1] \to H$ by

$$\gamma(s) := \Psi^{-1}\left(\frac{x_1 + s(x_2 - x_1)}{\|x_1 + s(x_2 - x_1)\|}\right).$$

Since $s \mapsto \left(\frac{x_1+s(x_2-x_1)}{\|x_1+s(x_2-x_1)\|}\right)$ is C^p, so is $\gamma(s)$. We first want to show that

$$0 < \lim_{s \to 0^+} \frac{f(\Psi \circ \gamma(s)) - f(x_1)}{\|\Psi \circ \gamma(s) - x_1\|}.$$

Since $\|x_1 + s(x_2 - x_1)\| \le 1$ for $s \in [0, 1]$, it follows that

$$sf(x_2 - x_1) = f(x_1 + s(x_2 - x_1)) - f(x_1) \le f(\Psi \circ \gamma(s)) - f(x_1).$$

Put $z = s(x_2 - x_1)$, and we have

$$\begin{aligned} \|\Psi \circ \gamma(s) - x_1\| &= \left\|\tfrac{x_1+z-x_1\|x_1+z\|}{\|x_1+z\|}\right\| = \left\|\tfrac{(x_1+z)(1-\|x_1+z\|)+z\|x_1+z\|}{\|x_1+z\|}\right\| \\ &\le |1 - \|x_1 + z\|| + \|z\| \le 2\|z\| = 2s\|x_2 - x_1\|. \end{aligned}$$