J. Math. Anal. Appl. 445 (2017) 1390–1393

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Removable singularities in C*-algebras of real rank zero

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ARTICLE INFO

Article history:
Received 12 November 2015
Available online 27 January 2016
Submitted by D. Ryabogin

Dedicated to Richard Aron with gratitude

Keywords:
Infinite dimensional holomorphy
Weak (FU)

ABSTRACT

Let $\mathfrak{A}$ be a C*-algebra with identity and real rank zero. Suppose a complex-valued function is holomorphic and bounded on the intersection of the open unit ball of $\mathfrak{A}$ and the identity component of the set of invertible elements of $\mathfrak{A}$. We give a short transparent proof that the function has a holomorphic extension to the entire open unit ball of $\mathfrak{A}$. The author previously deduced this from a more general fact about Banach algebras.

1. Preliminary definitions and theorems

Recall [1] that a C*-algebra is a closed complex subalgebra $\mathfrak{A}$ of the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators on a Hilbert space with the operator norm such that $\mathfrak{A}$ contains the adjoints of each of its elements. All our C*-algebras contain the identity operator I.

To give a basic example, let S be a compact Hausdorff space and let $C(S)$ be the algebra of all continuous complex-valued functions on S with the sup norm. Then there exist a Hilbert space H, a C*-algebra $\mathfrak{A}$ in $\mathcal{B}(H)$ and an isomorphism $\rho : C(S) \to \mathfrak{A}$ that preserves norms and adjoints. To see this, let H be the Hilbert space having the same dimension as the cardinality of S and let $\{e_s : s \in S\}$ be an orthonormal basis for H. Then we may take $\rho(f)$ to be the multiplication operator defined by $\rho(f)(e_s) = f(s)e_s$ for all $s \in S$ and $f \in C(S)$.

More generally, one can define a Banach algebra that is an abstraction of a C*-algebra and show that an isomorphism like the above exists. Specifically, a B*-algebra is a complex Banach algebra A with an involution * such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Then a norm and adjoint preserving isomorphism ρ of A onto a C*-algebra exists by the Gelfand–Naimark theorem [1, p. 209].

We now turn to some basic facts about complex-valued holomorphic functions defined on a domain D in a complex Banach space X. We say that a function $f : D \to \mathbb{C}$ is holomorphic if for each $x \in D$ there exists a continuous complex-linear functional $\ell \in X^*$ such that

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http://dx.doi.org/10.1016/j.jmaa.2016.01.053
0022-247X/

$$\lim_{y\to 0} \frac{f(x+y) - f(x) - \ell(y)}{\|y\|} = 0.$$

Clearly, if f is holomorphic in D then the function $\phi(\lambda) = f(x + \lambda y)$ is holomorphic (in the usual sense) in a neighborhood of the origin for each $x \in D$ and $y \in X$. It is well known [7, Theorem 3.17.1] that this property also implies holomorphy when f is locally bounded in D. One can extend many classical results about holomorphic functions by applying the above property. For example, this is true for the following elementary form of the identity theorem [7, Theorem 3.16.4].

Proposition 1. *Let D be a domain in a complex Banach space X and let $f : D \to \mathbb{C}$ be holomorphic in D. If f vanishes on a ball in D then f vanishes everywhere in D.*

By definition, a ball is a set of the form

$$B_r(x_0) = \{x \in X : \ \|x - x_0\| < r\},$$

where $x_0 \in X$ and $r > 0$.

We will need the following elementary version of Taylor's theorem, which can be proved as in [7, Theorem 3.17.1], and a simple converse, which can be obtained from the Weierstrass M-test and [7, Theorem 3.18.1].

Proposition 2. *Let X be a complex Banach space and let $x_0 \in X$ and $r > 0$. If $f : B_r(x_0) \to \mathbb{C}$ is a bounded holomorphic function, then for each n there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree n such that*

$$f(x) = \sum_{n=0}^{\infty} P_n(x - x_0) \quad \textit{for } x \in B_r(x_0). \tag{1}$$

Conversely, if for each n there is a continuous complex-homogeneous polynomial $P_n : X \to \mathbb{C}$ of degree n and if

$$\|P_n\| \le \frac{M}{r^n}, \quad n = 0, 1, \ldots \tag{2}$$

for some positive constants r and M, then the function f given by (1) is holomorphic in $B_r(x_0)$.

For example, if (1) holds then

$$P_n(y) = \frac{1}{n!} \frac{d^n}{dt^n} f(x_0 + ty)\bigg|_{t=0}, \quad n = 0, 1, \ldots \tag{3}$$

for all $y \in X$. If f is holomorphic on $B_r(x_0)$ and M is a bound for f, then (2) is a consequence of the classical Cauchy estimates. As usual,

$$\|P_n\| = \sup\{|P_n(x)| : \ \|x\| \le 1, x \in X\}.$$

2. Real rank zero

Definition 1. (See [2].) Let $\mathfrak{A}$ be a C*-algebra and let $\mathcal{S}$ be the set of self-adjoint elements of $\mathfrak{A}$. Then $\mathfrak{A}$ has *real rank zero* if the elements of $\mathcal{S}$ with finite spectra are dense in $\mathcal{S}$.