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C^*-convexity of norm unit balls

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This paper is dedicated to Professor Richard Martin Aron

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We characterize the norms on $\mathbb{B}(\mathcal{H})$ whose unit balls are C^*-convex. We call such norms M-norms and investigate their dual norms, named L-norms. We show that the class of L-norms greater than a given norm enjoys a minimum element and the class of M-norms less than a given norm enjoys a maximum element. These minimum and maximum elements will be determined in some cases. Finally, we give a constructive result to obtain M-norms and L-norms on $\mathbb{B}(\mathcal{H})$.

1. Introduction

Throughout this paper $\mathbb{B}(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I denotes the identity operator on $\mathcal{H}$. If $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with $\mathbb{M}_n$, the C^*-algebra of all $n \times n$ matrices with complex entries. We mean by $\mathbb{L}^1(\mathcal{H})$ the $*$-algebra of all trace class operators on $\mathcal{H}$. It is well-known that $\mathbb{B}(\mathcal{H})$ is identified with $\mathbb{L}^1(\mathcal{H})^*$, the dual space of $\mathbb{L}^1(\mathcal{H})$, under the mapping $S \mapsto \mathrm{Tr}(\cdot\, S)$.

A subset $\mathscr{K}$ of $\mathbb{B}(\mathcal{H})$ is called C^*-convex if $A_1, \ldots, A_k \in \mathscr{K}$ and $C_1, \ldots, C_k \in \mathbb{B}(\mathcal{H})$ with $\sum_{i=1}^k C_i^* C_i = I$ implies that $\sum_{i=1}^k C_i^* A_i C_i \in \mathscr{K}$. This kind of convexity has been introduced by Loebl and Paulsen [9] as a non-commutative generalization of the usual linear convexity. For example the sets $\{T \in \mathbb{B}(\mathcal{H});\ 0 \le T \le I\}$ and $\{T \in \mathbb{B}(\mathcal{H});\ \|T\| \le M\}$ are C^*-convex. It is evident that the C^*-convexity of a set $\mathscr{K}$ in $\mathbb{B}(\mathcal{H})$ implies its convexity in the usual sense. The converse is not however true in general. Various examples and some basic properties of C^*-convex sets are presented in [9].

In recent decades, many operator algebraists paid their attention to extend various concepts in the commutative setting to non-commutative cases. Regarding these works, the notion of C^*-convexity in

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C^*-algebras has been established as a non-commutative generalization of the linear convexity in linear spaces. Some essential results of convexity theory have been generalized in [3] to C^*-convexity. In particular, a version of the so-called Hahn–Banach theorem is presented. The operator extension of extreme points, the C^*-extreme points have also been introduced and studied, see [4,5,7,9]. Moreover, Magajna [10] established the notion of C^*-convexity for operator modules and proved some useful separation theorems. We refer the reader to [8,12,11,13] for further results concerning C^*-convexity.

The main aim of the present paper is to characterize the norms whose unit balls are C^*-convex.

In Section 2, we introduce M-norms and L-norms and give their properties and investigate some connections between them.

In Section 3, we show that the class of L-norms which are greater than an arbitrary norm $\|\cdot\|$, enjoys a minimum element and the class of M-norms which are less than an arbitrary norm $\|\cdot\|$, possesses a minimum element. We determine these minimum and maximum elements in some cases. For example, we will show that the trace norm $\|\cdot\|_1$ is the minimum element in the class of L-norms greater than the operator norm $\|\cdot\|_\infty$.

In Section 4, we give a constructive result to obtain M-norms.

2. M-type and L-type norms

We begin this section with the definitions of M-norm and L-norm.

Definition 2.1. A norm $\|\cdot\|$ on $\mathbb{B}(\mathcal{H})$ is an M-norm (is of M-type) if

$$\left\|\sum_{i=1}^{k} C_i^* X_i C_i\right\| \leq \max_{1\leq i\leq k} \|X_i\| \qquad \left(X_i \in \mathbb{B}(\mathcal{H}), \qquad \sum_{i=1}^{k} C_i^* C_i = I\right). \tag{1}$$

The motivation of the next definition will be revealed soon.

Definition 2.2. A norm $\|\cdot\|$ on $\mathbb{B}(\mathcal{H})$ is an L-norm (is of L-type) if

$$\sum_{i=1}^{k} \|C_i X C_i^*\| \leq \|X\| \qquad \left(X \in \mathbb{B}(\mathcal{H}), \quad \sum_{i=1}^{k} C_i^* C_i = I\right). \tag{2}$$

The next lemma is easy to prove. So we omit its proof.

Lemma 2.3. *Let $\|\cdot\|$ be a norm on $\mathbb{B}(\mathcal{H})$. The unit ball of $\|\cdot\|$ is C^*-convex if and only if $\|\cdot\|$ is a M-norm.*

It is well-known that $\{T \in \mathbb{B}(\mathcal{H});\ \|T\| \leq 1\}$ is C^*-convex. So, the operator norm is a M-norm.

The dual space of $\mathbb{M}_n$ is identified with $\mathbb{M}_n$ itself under the duality coupling

$$\langle Y|X\rangle = \mathrm{Tr}\,(Y^*X)\,, \quad X, Y \in \mathbb{M}_n.$$

Here, it should be noticed that

$$\langle Y|Z^*XZ\rangle = \langle ZYZ^*|X\rangle \qquad (X, Y, Z \in \mathbb{M}_n). \tag{3}$$

The dual norm $\|\cdot\|_*$ of a norm $\|\cdot\|$ is defined by

$$\|Y\|_* = \sup\{|\langle Y|X\rangle|; \quad \|X\| \leq 1\}\,, \quad \text{for all} \ \ Y \in \mathbb{M}_n. \tag{4}$$

A connection between M-norms and L-norms can be stated as follows.