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On a functional equation related to a pair of hedgehogs with congruent projections

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ABSTRACT

Hedgehogs are geometrical objects that describe the Minkowski differences of arbitrary convex bodies in the Euclidean space $\mathbb{E}^n$. We prove that two hedgehogs in $\mathbb{E}^n$, $n \geq 3$, coincide up to a translation and a reflection in the origin, provided that their projections onto any two-dimensional plane are directly congruent and have no direct rigid motion symmetries. Our result is a consequence of a more general analytic statement about the solutions of a functional equation in which the support functions of hedgehogs are replaced with two arbitrary twice continuously differentiable functions on the unit sphere.

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1. Introduction

In this paper we address several questions related to the following open problem (cf. [3], Problem 3.2, p. 125):

Problem 1. Suppose that $2 \leq k \leq n-1$ and that K and L are convex bodies in $\mathbb{E}^n$ such that the projection $K|H$ is directly congruent to $L|H$ for all subspaces H in $\mathbb{E}^n$ of dimension k. Is K a translate of $\pm L$?

Here, we say that two sets A and B in the Euclidean space $\mathbb{E}^k$ are *directly congruent* if there exists a rotation $\phi \in SO(k)$, such that $\phi(A)$ is a translate of B.

We refer the reader to [5], [3] (pp. 100–110), [6] (pp. 126–127), [13,15,2] for history and partial results related to this problem. In particular, V.P. Golubyatnikov considered Problem 1 in the case $k = 2$ and obtained the following result.

Theorem 1.1 *([5], Theorem 2.1.1, p. 13). Consider two convex bodies K and L in $\mathbb{E}^n$, $n \geq 3$. Assume that their projections on any two-dimensional plane passing through the origin are directly congruent and have no direct rigid motion symmetries, then $K = L + b$ or $K = -L + b$ for some $b \in \mathbb{E}^n$.*

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Here a set $A \subset \mathbb{E}^2$ has a direct rigid motion symmetry if it is directly congruent to itself.

In this paper we study a functional equation related to Problem 1 in the case $k = 2$. To formulate our main result we define an analogue of the notion of a direct rigid motion symmetry for functions on the unit circle S^1 in $\mathbb{E}^2$. We say that a function h on S^1 satisfies a *direct rigid motion symmetry equation* if there exists a non-trivial rotation $\phi \in SO(2)$ and $a \in \mathbb{E}^2$, such that

$$h(\phi(u)) + a \cdot u = h(u) \quad \text{for any} \quad u \in S^1. \tag{1}$$

Our main result is

Theorem 1.2. *Let f and g be two twice continuously differentiable real-valued functions on $S^{n-1} \subset \mathbb{E}^n$, $n \geq 3$. Assume that for any 2-dimensional plane α passing through the origin there exists a vector $a_\alpha \in \alpha$ and a rotation $\phi_\alpha \in SO(2,\alpha)$, such that the restrictions of f and g onto the large circle $S^{n-1} \cap \alpha$ satisfy the equation*

$$f(\phi_\alpha(u)) + a_\alpha \cdot u = g(u) \quad \forall u \in S^{n-1} \cap \alpha. \tag{2}$$

Then there exists $b \in \mathbb{E}^n$ such that for all $u \in S^{n-1}$ we have $g(u) = f(u) + b \cdot u$ or $g(u) = f(-u) + b \cdot u$, provided that the restrictions of f, g onto any such large circle $S^{n-1} \cap \alpha$ do not satisfy the direct rigid motion symmetry equation.

If f and g are the support functions of convex bodies K and L in $\mathbb{E}^n$, $n \geq 3$, respectively, we reproduce the aforementioned result of V.P. Golubyatnikov, [5]. Our approach is based on his ideas together with an application of the connection between twice continuously differentiable functions on the unit sphere and support functions of convex bodies. It allows, in particular, to get rid of the convexity assumption on functions.

In the case when the orthogonal transformations ϕ_ξ degenerate into identity or reflection with respect to the origin, we show that the assumptions on the lack of symmetries and smoothness are not necessary. We have

Theorem 1.3. *Let $2 \leq k \leq n-1$ and let f, g be two continuous real-valued functions on $S^{n-1} \subset \mathbb{E}^n$. Assume that for any k-dimensional plane α passing through the origin and some vector $a_\alpha \in \alpha$, the restrictions of f and g onto $S^{n-1} \cap \alpha$ satisfy at least one of the equations*

$$f(-u) + a_\alpha \cdot u = g(u) \quad \textit{for all} \quad u \in \alpha \cap S^{n-1}, \ \textit{or}$$
$$f(u) + a_\alpha \cdot u = g(u) \quad \textit{for all} \quad u \in \alpha \cap S^{n-1}.$$

Then there exists $b \in \mathbb{E}^n$ such that for all $u \in S^{n-1}$ we have $g(u) = f(u) + b \cdot u$ or $g(u) = f(-u) + b \cdot u$.

As one of the applications of Theorem 1.2 we also obtain a result about the classical hedgehogs, which are geometrical objects that describe the Minkowski differences of arbitrary convex bodies in $\mathbb{E}^n$.

The idea of using Minkowski differences of convex bodies may be traced back to some papers by A.D. Alexandrov and H. Geppert in the 1930's (see [1,4]). Many notions from the theory of convex bodies carry over to hedgehogs and quite a number of classical results find their counterparts (see, for instance, [10]). Classical hedgehogs are (possibly singular, self-intersecting and non-convex) hypersurfaces that describe differences of convex bodies with twice continuously differentiable support functions in $\mathbb{E}^n$. We refer the reader to works of Y. Martinez-Maure, [9,11,12], for more information on this topic.