J. Math. Anal. Appl. 445 (2017) 1516–1529

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Inequalities for generalized Euclidean operator radius via Young's inequality

Alemeh Sheikhhosseini[a], Mohammad Sal Moslehian[b,*], Khalid Shebrawi[c]

[a] *Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*
[b] *Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran*
[c] *Department of Mathematics, Al-Balqa' Applied University, Salt, Jordan*

ARTICLE INFO

Article history:
Received 19 November 2015
Available online 1 April 2016
Submitted by D. Ryabogin

This paper is dedicated to Professor Richard M. Aron

Keywords:
Young inequality
Generalized Euclidean operator radius
Inequality
Numerical range

ABSTRACT

Using a refinement of the classical Young inequality, we refine some inequalities of operators including the function ω_p, where ω_p is defined for $p \geqslant 1$ and operators $T_1, \ldots, T_n \in \mathbb{B}(\mathscr{H})$ by

$$\omega_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^{n} |\langle T_i x, x\rangle|^p \right)^{\frac{1}{p}}.$$

Among other things, we show that if $T_1, \ldots, T_n \in \mathbb{B}(\mathscr{H})$ and $p \geq q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\frac{1}{n} \|\sum_{i=1}^{n} T_i\|^2 \leq \omega_p(|T_1|, \ldots, |T_n|)\omega_q(|T_1^*|, \ldots, |T_n^*|)$$
$$\leq \frac{1}{p}\|\sum_{i=1}^{n} |T_i|^p\| + \frac{1}{q}\|\sum_{i=1}^{n} |T_i^*|^q\| - \inf_{\|x\|=\|y\|=1} \delta(x, y),$$

where $\delta(x, y) = \frac{1}{p}(\sqrt{\sum_{i=1}^{n} \langle |T_i|x, x\rangle^p} - \sqrt{\sum_{i=1}^{n} \langle |T_i^*|y, y\rangle^q})^2$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|.\|$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with

* Corresponding author.
E-mail addresses: sheikhhosseini@uk.ac.ir (A. Sheikhhosseini), moslehian@um.ac.ir (M.S. Moslehian), khalid@bau.edu.jo (K. Shebrawi).

http://dx.doi.org/10.1016/j.jmaa.2016.03.079

the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. The numerical range of an operator $A \in \mathbb{B}(\mathscr{H})$ is defined by $W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H}, \|x\| = 1\}$. For any $A \in \mathbb{B}(\mathscr{H})$, $\overline{W(A)}$ is a convex subset of the complex plane containing the spectrum of A; see [3,4,14] for more information.

The numerical radius of $A \in \mathbb{B}(\mathscr{H})$ is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

It is known that $\omega(\cdot)$ is a norm on $\mathbb{M}_n$, but it is not unitarily invariant. The quantity $\omega(A)$ is useful in studying perturbation, convergence and approximation problems as well as iterative method, etc. For more information see [2,9].

For positive real numbers a, b, the classical Young inequality says that if $p, q > 1$ such that $1/p+1/q = 1$, then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

A refinement of the scalar Young inequality is presented in [8] as follows:

$$ab + r_0(a^{p/2} - b^{q/2})^2 \le \frac{a^p}{p} + \frac{b^q}{q}, \tag{1.1}$$

where $r_0 = \min\{1/p, 1/q\}$. Recently, Al-Manasrah and Kittaneh [1] generalized inequality (1.1) to

$$\left(a^{\frac{1}{p}} b^{\frac{1}{q}}\right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \le \left(\frac{a}{p} + \frac{b}{q}\right)^m, \tag{1.2}$$

where $m = 1, 2, \ldots$; see also [12]. Furthermore, it is known that for $r \ge 1$,

$$\left(\frac{a}{p} + \frac{b}{q}\right)^m \le \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}. \tag{1.3}$$

Thus

$$\left(a^{\frac{1}{p}} b^{\frac{1}{q}}\right)^m + r_0^m \left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \le \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}. \tag{1.4}$$

In particular, if $p = q = 2$, then

$$\left(a^{\frac{1}{2}} b^{\frac{1}{2}}\right)^m + \frac{1}{2^m}(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \le 2^{\frac{-m}{r}} \left(a^r + b^r\right)^{\frac{m}{r}}. \tag{1.5}$$

Let $T_1, \ldots, T_n \in \mathbb{B}(\mathscr{H})$. The Euclidean operator radius of $T_1, \ldots, T_n$ is defined in [11] by

$$\omega_e(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^n |\langle T_i x, x\rangle|^2\right)^{\frac{1}{2}}.$$

In addition, the functional ω_p of operators $T_1, \ldots, T_n$ for $p \geqslant 1$ is defined in [13] by

$$\omega_p(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{i=1}^n |\langle T_i x, x\rangle|^p\right)^{\frac{1}{p}}.$$

The authors of [13] obtained some inequalities for $\omega_p(B, C)$ of two bounded linear operators in $\mathbb{B}(\mathscr{H})$ and found some upper bounds for $\omega_p(T_1, \ldots, T_n)$.