J. Math. Anal. Appl. 445 (2017) 125–138

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

An ergodic value distribution of certain meromorphic functions ☆

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ARTICLE INFO

Article history:
Received 16 March 2016
Available online 2 August 2016
Submitted by Richard M. Aron

Keywords:
Lindelöf hypothesis
Birkhoff's ergodic theorem
Zeta function
L-function
Derivative

ABSTRACT

We calculate a certain mean-value of meromorphic functions by using specific ergodic transformations, which we call affine Boolean transformations. We use Birkhoff's ergodic theorem to transform the mean-value into a computable integral which allows us to completely determine the mean-value of this ergodic type. As examples, we introduce some applications to zeta functions and L-functions. We also prove an equivalence of the Lindelöf hypothesis of the Riemann zeta function in terms of its certain ergodic value distribution associated with affine Boolean transformations.

1. Introduction

In [7], M. Lifshits and M. Weber investigated the value distribution of the *Riemann zeta function* $\zeta(s)$ by using the Cauchy random walk. They proved that almost surely

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\zeta\left(\frac{1}{2}+iS_n\right)=1+o\left(\frac{(\log N)^b}{N^{1/2}}\right)$$

holds for any $b>2$ where $\{S_n\}_{n=1}^{\infty}$ is the Cauchy random walk. This result implies that most of the values of $\zeta(s)$ on the critical line are quite small. Analogous to [7], T. Srichan investigated the value distributions of Dirichlet L-functions and Hurwitz zeta functions by using the Cauchy random walk in [9].

The first approach to investigate the ergodic value distribution of $\zeta(s)$ was done by J. Steuding. In [11], he studied the ergodic value distribution of $\zeta(s)$ on vertical lines under the Boolean transformation. The notion of Boolean transformation comes from the following formula due to Boole

☆ The first named author is supported by JSPS KAKENHI Grant Number 16J01139. The second named author is supported by JSPS KAKENHI Grant Number 15J02325.

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http://dx.doi.org/10.1016/j.jmaa.2016.07.064
0022-247X/

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx$$

which is valid for any integrable function f and was later studied by Glaisher. We refer to (1) in R.L. Adler and B. Weiss [1].

We are interested in studying the ergodic value distribution of a larger class of meromorphic functions which includes but is not limited to the Selberg class (of ζ-functions and L-functions) and their derivatives, on vertical lines under more general Boolean transformations, which we shall call *affine Boolean transformation* $T_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ given by

$$T_{\alpha,\beta}(x) := \begin{cases} \frac{\alpha}{2}\left(\frac{x+\beta}{\alpha} - \frac{\alpha}{x-\beta}\right), & x \neq \beta; \\ \beta, & x = \beta \end{cases} \tag{1.1}$$

for an $\alpha > 0$ and a $\beta \in \mathbb{R}$. Below is our main theorem. For a given $c \in \mathbb{R}$, we shall denote by $\mathbb{H}_c$ and $\mathbb{L}_c$ the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > c\}$ and the line $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = c\}$.

Theorem 1.1. *Let f be a meromorphic function on $\mathbb{H}_c$ satisfying the following conditions.*

(1) *There exists an $M > 0$ and a $c' > c$ such that for any $t \in \mathbb{R}$, we have*

$$|f(\{\sigma + it \mid \sigma > c'\})| \leq M.$$

(2) *There exists a non-increasing continuous function $\nu : (c, \infty) \to \mathbb{R}$ such that if σ is sufficiently near c then $\nu(\sigma) \leq 1 + c - \sigma$, and that for any small $\epsilon > 0$, $f(\sigma + it) \ll_{f,\epsilon} |t|^{\nu(\sigma)+\epsilon}$ as $|t| \to \infty$.*

(3) *f has at most one pole of order m in $\mathbb{H}_c$ at $s = s_0 = \sigma_0 + it_0$, that is, we can write its Laurent expansion near $s = s_0$ as*

$$\frac{a_{-m}}{(s-s_0)^m} + \frac{a_{-(m-1)}}{(s-s_0)^{m-1}} + \cdots + \frac{a_{-1}}{s-s_0} + a_0 + \sum_{n=1}^{\infty} a_n (s-s_0)^n \tag{1.2}$$

for $m \geq 0$, where we set $m = 0$ if f has no pole in $\mathbb{H}_c$.

Then for any $s \in \mathbb{H}_c \backslash \mathbb{L}_{\sigma_0}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f\left(s + iT_{\alpha,\beta}^n x\right) = \frac{\alpha}{\pi} \int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2 + (\tau-\beta)^2} d\tau \tag{1.3}$$

for almost all $x \in \mathbb{R}$.

We denote the right-hand side of the above formula by $l_{\alpha,\beta}(s)$. If f has no pole in $\mathbb{H}_c$,

$$l_{\alpha,\beta}(s) = f(s + \alpha + i\beta) \tag{1.4}$$

for all $s \in \mathbb{H}_c$. If f has a pole at $s = s_0 = \sigma_0 + it_0$,

$$l_{\alpha,\beta}(s) = \begin{cases} f(s+\alpha+i\beta) + B_m(s_0), & c < \mathrm{Re}(s) < \sigma_0, s \neq s_0 - \alpha - i\beta; \\ \sum_{n=0}^{m} \frac{a_{-n}}{(-2\alpha)^n}, & c < \mathrm{Re}(s) < \sigma_0, s = s_0 - \alpha - i\beta; \\ f(s+\alpha+i\beta), & \mathrm{Re}(s) > \sigma_0; \end{cases} \tag{1.5}$$