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On the structure of C_0-semigroups of holomorphic Carathéodory isometries in Hilbert space

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We establish closed formulas for all strongly continuous one-parameter semigroups of holomorphic Carathéodory isometries of the unit ball of a Hilbert space in terms of spectral resolutions of skew self-adjoint dilations related to the Reich–Shoikhet nonlinear infinitesimal generator.

1. Introduction

Throughout this work $\mathbf{H}$ denotes a fixed complex Hilbert space with scalar product $\langle x|\, y\rangle$ which is linear (i.e. $\mathbb{C}$-linear) in x and antilinear in y and we shall write $\|x\| := \langle x|x\rangle^{1/2}$ for the canonical norm. We use the notations $\mathbf{B} := \{x \in \mathbf{H} : \|x\| < 1\}$, $a^* := [x \mapsto \langle x|a\rangle]$ for the open unit ball, and the adjoint representation of bounded linear functionals, respectively. We regard the elements h, h^* $(h \in \mathbf{H})$ as column resp. row matrices and, given a linear map $A : \mathbf{S} \to \mathbf{H}$ on some linear submanifold of $\mathbf{H}$, we apply the canonical $\mathbf{H} \oplus \mathbb{C}$-split matrix identifications $x \oplus \xi \equiv \left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right]$ resp. $\left[\begin{smallmatrix} A & b \\ c^* & d \end{smallmatrix}\right] \equiv \big[x \oplus \xi \mapsto (Ax + b) \oplus (c^*x + d)\big]$ with $x \in \mathbf{S}$, $b, c \in \mathbf{H}$ and $\xi, d \in \mathbb{C}$. This gives rise to the familiar linear representation of fractional linear maps on $\mathbf{H}$:

$$\mathfrak{F}\left(\begin{bmatrix} A & b \\ c^* & d \end{bmatrix}\right) := \big[x \mapsto (c^*x + d)^{-1}(Ax + b)\big].$$

Our object of chief interest will be the semigroup $\mathrm{Iso}(d_{\mathbf{B}})$ of all holomorphic isometries of $\mathbf{B}$ with respect to the Carathéodory metric $d_{\mathbf{B}}$. Recall [3,4] that all its elements are fractional linear maps, namely they are compositions of *Möbius transformations*[1] with linear isometries of $\mathbf{H}$ (restricted to $\mathbf{B}$). In 1987, in his pioneering work [12], Vesentini established that the correspondence

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[1] Fractional linear transformations mapping $\mathbf{B}$ injectively onto itself.

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$$\mathfrak{F}^{\#} : \big[\mathcal{U}^t : t\in\mathbb{R}_+\big] \mapsto \big[\mathfrak{F}(\mathcal{U}^t)|\mathbf{B} : \ t\in\mathbb{R}_+\big]$$

maps the family $\mathcal{C}_0\mathcal{S}\big(\mathrm{Iso}(\mathbf{H})\big)$ of all strongly continuous one-parameter semigroups of linear isometries of the indefinite norm $\|x\|^2 - |\xi|^2$ on $\mathbf{H}\oplus\mathbb{C}$ into the family $\mathcal{C}_0\mathcal{S}\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ of all strongly continuous one-parameter semigroups $[\Psi^t : t\in\mathbb{R}_+]\subset\mathrm{Iso}(d_{\mathbf{B}})$.[2] According to [12, Th.VII], given $[\mathcal{U}^t : t\in\mathbb{R}_+]\in\mathfrak{S}$ with the infinitesimal generator $\mathcal{A} = \frac{d}{dt}\big|_{t=0+}\mathcal{U}^t$, for the corresponding non-linear objects $\Psi^t := \mathfrak{F}(\mathcal{U}^t)\big|\mathbf{B}$ we have $\big\{p\in\mathbf{B} : t\mapsto \Psi^t(p) \text{ is differentiable}\big\} = \big\{x\in\mathbf{B} : \ x\oplus 1\in\mathrm{dom}(\mathcal{A})\big\}$, and the latter set is dense in the ball $\mathbf{B}$. It is well known [12,5] that here we can identify the linear operator $\mathcal{A}$ (which is densely defined in $\mathbf{H}\oplus\mathbb{C}$) with an $\mathbf{H}\oplus\mathbb{C}$-split matrix if and only if the orbit $t\mapsto\Psi^t(0)$ is differentiable. This happens if and only if the generator $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{bmatrix} iA+\nu & b \\ b^* & \nu \end{bmatrix}, \qquad \nu\in\mathbb{C},\ b\in\mathbf{H},\ A\in\mathrm{Her_s}(\mathbf{H}) \tag{1.1}$$

with $\mathrm{dom}(\mathcal{A}) = \mathrm{dom}(A)\oplus\mathbb{C}$ where $\mathrm{Her_s}(\mathbf{H})$ stands for the family of all unbounded $\mathbf{H}$-*hermitian* operators (maximal symmetric closed linear operators with dense domain in $\mathbf{H}$). Even the cases with non-differentiable 0-orbit can be treated by passing to a semigroup $[\Phi^t : t\in\mathbb{R}_+]$ of the form $\Theta^{-1}\circ\Psi^t\circ\Theta$ with any Möbius transformation Θ such that $\Theta(0)\in\mathrm{dom}\big(\frac{d}{dt}\big|_{t=0+}\Psi^t(0)\big)$. Since the Möbius group is transitive on $\mathbf{B}$, hence any strongly continuous one-parameter semigroup $[\Psi^t : t\in\mathbb{R}_+]\in\mathcal{C}_0\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ is equivalent up to a Möbius transformation (*Möbius equivalent* for short in the sequel) to a semigroup $[\Phi^t : t\in\mathbb{R}_+]\in\mathcal{C}_0\big(\mathrm{Iso}(d_{\mathbf{B}})\big)$ whose infinitesimal generator [8,9] has the form

$$\Gamma(x) = \frac{d}{dt}\Big|_{t=0+}\Phi^t \ = b - \langle x|b\rangle x + iAx, \quad x\in\mathrm{dom}(R)\cap\mathbf{B} \tag{1.2}$$

with some maximal symmetric operator A defined densely on $\mathbf{H}$ and some vector $b\in\mathbf{H}$. Also conversely, if iA is the infinitesimal generator for some strongly continuous one-parameter subsemigroup of $\mathcal{L}(\mathbf{H})$ then, for any $b\in\mathbf{H}$, the vector field (1.2) is the infinitesimal generator of a strongly continuous one-parameter subsemigroup of $\mathrm{Iso}(d_{\mathbf{B}})$. It is worth noticing that Kaup [6,7] achieved a far-reaching Jordan-theoretical analog of (1.2) describing the complete holomorphic vector fields of the unit ball of JB*-triple and integrated them for the case $A=0$ resulting in a fractional linear type formula for generalized Möbius transformations. However, strong continuity destroys such an elegant setting. In [13,5] these considerations were extended to semigroups of fractional linear transformations arising from a strongly continuous one-parameter semigroup applied to the solutions of Ricatti type equations $\dot{x} = \Gamma(x)$ with vector fields analogous to (1.2) in reflexive Hilbert C^*-modules, but without providing explicit algebraic formulas.

2. Results

Henceforth, for short, C_0-*semigroup* [resp. C_0-*group*] will mean *strongly continuous one-parameter semigroup* [-*group*]. We shall write $\mathrm{gen}[U^t : t\in\mathbb{R}_+]$ or $\mathrm{gen}[\widetilde{U}^t : t\in\mathbb{R}]$ for the infinitesimal generator of the C_0-semigroup $[U^t : t\in\mathbb{R}_+]$ or C_0-group $[\widetilde{U}^t : t\in\mathbb{R}]$, respectively. Given a closed subspace $\mathbf{K}$ in the Hilbert spaces $\mathbf{H}$ or $\mathbf{H}\oplus\mathbb{C}$, let $\mathrm{P}_{\mathbf{K}}$ be the orthogonal projection onto $\mathbf{K}$ (without danger of confusion).

In this paper we develop a triangularization method leading to explicit algebraic formulas for a C_0-semigroup generated by a vector field (1.2). This will be done in terms of fixed points of Γ and quadratures of a C_0-semigroup formed by complex linear isometries of a suitable 1-codimensional subspace of $\mathbf{H}$.

[2] It seems that so far no argument appeared in the literature concerning the plausible surjectivity of the map $\mathfrak{F}^{\#}$. The question is rather harmless in our setting: in the case of the unit ball of a Hilbert space an argument with joint fixed points (Lemma 3.1) furnishes a positive answer. However, e.g. in the case of the unit ball of $\mathcal{L}(\mathbf{H})$, the surjectivity of the respective $\mathfrak{F}^{\#}$ seems to be open and highly non-trivial.