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Norm-attaining property for a dual pair of Banach spaces

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A R T I C L E I N F O

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A B S T R A C T

In this study, we introduce the norm-attaining property for a dual pair of Banach spaces. We use this property to determine a quotient Banach space as a dual space. We also apply this property to obtain another proof of the James's characterization theorem regarding the reflexivity of a separable Banach space. In addition, the norm-attaining property of a Fourier space is studied.

1. Preliminaries

In this study, for a normed space E, we write $\|\cdot\|_E$ for its norm and B_E for its closed unit ball. In the following, let Q be a duality between two complex Banach spaces, X and Z, i.e., $Q : X \times Z \longrightarrow \mathbb{C}$ is a bounded bilinear map. In particular, when Z is a closed subspace of X^*, we always consider the natural duality between X and Z. Put $\theta : X \longrightarrow Z^*$ as the restriction map on X induced by Q, which is given by $x \in X \mapsto Q(x, \cdot) \in Z^*$. Z is said to be *separating* if θ is injective. In addition, Z is said to be *α-norming* for some $\alpha > 0$ if $\alpha\|x\|_X \le \|\theta(x)\|_{Z^*}$ for all $x \in X$.

According to previous studies, an element in the dual space of a Banach space is said to be *norm-attaining* if its norm is attained on the closed unit ball of the space (see [11]). In this study, we consider a more general setting. Using the notation given above, we say that Z has a *norm-attaining property* on a convex subset C of X with respect to Q (*or we simply state that Z has a norm-attaining property*) if for each element $z \in Z$, the real linear part $\mathcal{R}e\ Q(\cdot, z)$ attains the supremum on C and has the maximal value of $\|z\|_Z$, i.e., there is an element $x_0 \in C$ such that

$$\|z\|_Z = \sup\{\mathcal{R}e\ Q(x, z) : x \in C\} = \mathcal{R}e\ Q(x_0, z).$$

Similarly, the norm-attaining property of X can be defined via the duality Q.

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Let $B_Z^0 := \{x \in X : |Q(x,z)| \leq 1 \text{ for all } z \in B_Z\}$ denote the polar set of B_Z in X. Clearly, B_Z^0 is a closed convex balanced absorbing subset of X and $\theta(B_Z^0) \subseteq B_{Z^*}$. Moreover, if Z has a norm-attaining property on C, then $C \subseteq B_Z^0$.

The dual of a Banach space plays an important role in the study of Banach space theory. First, let us recall Godefroy's result. If F is a closed subspace of the dual space of a separable Banach space E, then F is α-norming and has a norm-attaining property on B_E if and only if the canonical map from E to F^* is an isometric isomorphism [6, Theorem VIII.2].

In this study, we generalize Godefroy's result to the case of a dual pair of Banach spaces (see Theorem 2.3). Using the notation given above, one of the consequences of our main result is that if X is separable and Z has a norm-attaining property on a convex subset of X (*which does need not to be the closed unit ball of* X), then the corresponding restriction θ on X has a dense range in Z^*. Thus, if θ has a closed range, then Z is a predual of a quotient of X in this case. One of the applications of this result is a new approach to the well-known James' characterization of the reflexivity of a separable Banach space case via a certain duality (see Corollary 2.6).

This study aims to investigate the norm-attaining property from an operator-theoretic aspect. If we let ρ be a representation of a C^*-algebra A and let $\mathcal{M}_*(\rho)$ be the predual of the von Neumann algebra generated by $\rho(A)$, then there is a natural duality between A and $\mathcal{M}_*(\rho)$ (see Section 3). In the final section, we study the case where A is the group C^*-algebra $C^*(G)$ of a second countable locally compact group G. Now, for a unitary representation π of G, it is known that the Fourier space $A_\pi(G)$ of G is the predual of the von Neumann algebra generated by $\pi(G)$. By applying our main result, we show that $A_\pi(G)$ has a norm-attaining property if and only if the canonical image of $C^*(G)$ under π is finite dimensional. Thus, we show that if the Fourier algebra $A(G)$ of G has a norm-attaining property, then G must be finite. A brief explanation of Fourier spaces is given in Section 3. Full details of Fourier spaces were given by Arsac [1].

2. Main results

We start with the following simple lemma, which can be obtained directly by the definition of polar sets and the w^*-compactness of the closed unit ball of a dual space.

Lemma 2.1. *Let X and Z be the Banach spaces with a duality Q. Then, the associated restriction map $\theta : X \longrightarrow Z^*$ is surjective if and only if $\theta(B_Z^0) = B_{Z^*}$. In this case, Z has a norm-attaining property on B_Z^0.*

Proposition 2.2. *Let d be a bounded linear surjection from a Banach space E onto a dual space F^* of some Banach space F. Let D be the associated duality between E and F, which is defined by $(x,y) \in E \times F \mapsto \langle y, d(x)\rangle$. Then, F has a norm-attaining property with respect to D.*

Proof. Note that the map $d : E \longrightarrow F^*$ is simply the restriction map on E with respect to the duality D defined by d. d is surjective, so Lemma 2.1 immediately implies that F has a norm-attaining property on B_F^0 with respect to the duality D. □

Now, we study some form of converse of Proposition 2.2. We use the following two known facts later.

(i) (Simons' inequality): Let D be any non-empty set and let M be a subset of D. If (f_n) is sequence of uniform bounded real-valued functions on D that satisfy the following condition: for any sequence $c_n \geq 0$ with $\sum c_n = 1$, $y \in M$ exists such that $\sum c_n f_n(y) = \sup_{x\in D} \sum c_n f_n(x)$, then we have $\sup_{y\in M} \overline{\lim_{n\to\infty}} f_n(y) \geq \overline{\lim_{n\to\infty}} f_n(x)$ for all $x \in D$ (see [15, Theorem 3]).

(ii) (Pryce's theorem): Suppose that Y is a topological space and $\mathcal{C}_p(Y)$ is the space of all real-valued continuous functions on Y endowed with pointwise convergence topology. Let A be a relative compact