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On Bohr's equivalence theorem

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A R T I C L E I N F O

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A B S T R A C T

In this note we show a converse of Bohr's equivalence theorem.

1. Introduction

Bohr's interest in the Riemann zeta function led him to study the set of values taken by Dirichlet series in their half plane of absolute convergence. For this problem Bohr developed a new method: associating to any Dirichlet series a power series with infinitely many variables (see [3]). He then introduced an equivalence relation among Dirichlet series and showed that equivalent Dirichlet series take the same set of values in certain open half planes (see [4]). We give here a very brief account of this theory; for a complete treatment we refer to Bohr's original work [4] and to Chapter 8 of Apostol [1].

We call *general Dirichlet series* any complex function $f(s)$, in the variable $s = \sigma + it$, that has a series representation of the form

$$f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s},$$

where the *coefficients* $a(n)$ are complex and the sequence of *exponents* $\Lambda = \{\lambda(n)\}$ consists of real numbers such that $\lambda(1) < \lambda(2) < \cdots$ and $\lambda(n) \to \infty$ as $n \to \infty$.

Note that this class of general Dirichlet series includes both power series, when $\lambda(n) = n$, and ordinary Dirichlet series, when $\lambda(n) = \log(n)$.

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Remark. The above definition of general Dirichlet series is the one that is given in the work of Bohr [4], and it is more restrictive then the usual definition, which is already present in later works of Bohr (see e.g. [5]). Restricting to the above setting has the advantage that the region of absolute convergence of the series is a right half plane, like for ordinary Dirichlet series (see e.g. [1, §8.2]). Since this is fundamental in the proof of the converse theorem, we have decided to work in the same setting of [4]. However, as we remarked in our Ph.D. thesis [8], Bohr's equivalence theorem holds true *mutatis mutandis* even for the usual general Dirichlet series.

Following Bohr (see e.g. [1, §8.3]), given a sequence of exponents $\Lambda = \{\lambda(n)\}$, we say that a sequence of real numbers $B = \{\beta(n)\}$ is a *basis* for Λ if it satisfies the following conditions:

(i) the elements of B are linearly independent over the rationals;
(ii) for every n, $\lambda(n)$ is expressible as a finite linear combination over $\mathbb{Q}$ of elements of B;
(iii) for every n, $\beta(n)$ is expressible as a finite linear combination over $\mathbb{Q}$ of elements of Λ.

We may express the above conditions in matrix notation by considering Λ and B as infinite column vectors (see [1, §8.4]). In particular, if B is a basis for Λ, we may write $\Lambda = RB$ and $B = T\Lambda$ for some *Bohr matrices* R and T.

We fix a sequence of exponents $\Lambda = \{\lambda(n)\}$ and a basis B of Λ, so that we may write $\Lambda = RB$. Consider two general Dirichlet series with the same sequence of exponents Λ, say

$$f(s) = \sum_{n=1}^{\infty} a(n)e^{-\lambda(n)s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} b(n)e^{-\lambda(n)s}.$$

Then, we say that $f(s)$ and $g(s)$ are *equivalent* ($f \sim g$), with respect to B, if there exists a sequence of real numbers $Y = \{y(n)\}$ such that

$$b(n) = a(n)e^{i(RY)_n} \qquad \text{for every } n. \tag{1}$$

We may now state Bohr's equivalence theorem (cf. Apostol [1, Theorem 8.16]), which is, roughly speaking, a combination of Kronecker's approximation theorem, Rouché's theorem and the absolute convergence of the Dirichlet series.

Theorem A *(Bohr, [4, Satz 4]). Let $f_1(s)$ and $f_2(s)$ be equivalent general Dirichlet series absolutely convergent for $\sigma > \alpha$. Then in any open half plane $\sigma > \sigma_0 \geq \alpha$ the functions $f_1(s)$ and $f_2(s)$ take the same set of values.*

Remark. Although Bohr's equivalence theorem is usually stated for open half planes, we have already remarked in [9] and [8] that Theorem A holds true also for open vertical strips.

In particular, one gets immediately the following more practical version of the above theorem, in the sense that in the applications on the value distribution of L-functions one usually appeals to this statement. Similar results in particular cases may be found for example in Bohr [2, §2], Titchmarsh [10, §11.4], Bombieri and Mueller [7, Lemma 1] and Bombieri and Ghosh [6, p. 240].

Theorem B. *Let $f(s)$ be a general Dirichlet series absolutely convergent for $\sigma > \alpha$, and let $V_f(\sigma_0)$ be the set of values taken by $f(s)$ on the vertical line $\sigma = \sigma_0 > \alpha$. Then*

$$\overline{V_f(\sigma_0)} = \{g(\sigma_0) \mid g \sim f\}.$$