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Reducing subspaces for a class of non-analytic Toeplitz operators on the bidisk ☆

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ABSTRACT

In this paper, we completely characterize all the reducing subspaces for a class of non-analytic Toeplitz operators with symbol $\varphi(z,w) = \alpha z^k + \beta \overline{w}^l$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha\beta \neq 0$. We also prove that the von Neumann algebra $\mathcal{V}^*(\varphi) = \{T_\varphi, T_\varphi^*\}'$ is abelian.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$ and $dA(z)$ denote the normalized area measure over $\mathbb{D}$. Let $A^2(\mathbb{D}^2)$ denote the Bergman space consisting of all holomorphic functions over $\mathbb{D}^2$, which are square integrable with respect to the normalized volume measure $dA(z)dA(w)$. Then $A^2(\mathbb{D}^2)$ is a Hilbert space with inner product $\langle f, g\rangle = \int_{\mathbb{D}^2} f\overline{g}dA(z)dA(w)$. Given an essentially bounded function ϕ, the Toeplitz operator T_ϕ is defined by $T_\phi f = P(\phi f)$ for $f \in A^2(\mathbb{D}^2)$. Put $\mathcal{V}^*(\phi) = \{T_\phi, T_\phi^*\}'$, the commutant algebra of the C^*-algebra generated by T_ϕ in $B(A^2(\mathbb{D}^2))$. As is given in [2], $\mathcal{V}^*(\phi)$ is a von Neumann algebra and is the norm closed linear span of its projections.

For a bounded linear operator S on a Hilbert space $\mathcal{H}$, a closed subspace $\mathcal{M}$ is called a reducing subspace for S if $S\mathcal{M} \subseteq \mathcal{M}$ and $S\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. In addition, $\mathcal{M}$ is called minimal if there is no nonzero reducing subspace $\mathcal{N}$ satisfying $\mathcal{N} \subsetneqq \mathcal{M}$. It is well known that $\mathcal{M}$ is a reducing subspace for S if and only if $SP_\mathcal{M} = P_\mathcal{M}S$, where $P_\mathcal{M}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. In this way, the range of projections in $\mathcal{V}^*(\phi)$ and the reducing subspaces for T_ϕ are in one-to-one correspondence. Therefore, in some sense,

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studying the structure of von Neumann algebra $\mathcal{V}^*(\phi)$ is equivalent to investigating the structure of the reducing subspaces for T_ϕ.

Let B_N denote a Blaschke product of finite order N on $\mathbb{D}$. In 2009, Zhu [20] proved that a multiplication operator M_{B_2} on $L_a^2(\mathbb{D})$ has two distinct nontrivial minimal reducing subspaces, and conjectured M_{B_N} has exactly N distinct nontrivial minimal reducing subspaces. In particular, if $B_N(z) = z^N$, M_{z^N} is a weighted unilateral shift operator of finite multiplicity on a weighted sequence space. Stessin and Zhu [16] showed that every reducing subspace for M_{z^N} contains a minimal reducing subspace as $X_n = \overline{\text{span}\{z^{n+kN} : k = 0, 1, 2, \cdots\}}$ with $0 \le n \le N-1$. What is worth mentioning, Hardy spaces, Bergman spaces and Dirichlet Spaces are three particular cases of the weighted sequence spaces. Further, Douglas and Kim [4], Li, Lan and Liu [13] generalized the results to some weighted unilateral shift operators on $L_a^2(A_r)$ and $F_\alpha^2(\alpha > 0)$ (the square integrable analytic functions on the annulus A_r with respect to the normalized measure $dA(z)$, and the square integrable entire functions on the whole complex plane $\mathbb{C}$ with respect to the Gaussian measure, respectively). In 2004, Hu, Sun, Xu and Yu [12] proved that there is always a nontrivial reducing subspace for M_{B_N}. In 2009, Guo, Sun, Zheng and Zhong [10] disproved Zhu's conjecture and proposed the modified conjecture that M_{B_N} has at most N distinct nontrivial minimal reducing subspaces. On the basis of the hard work (see [6,7,10,17,18], etc.) by Guo, Huang, Sun, Zheng and Zhong, et al., Douglas, Putinar and Wang [5] obtained that the number of nontrivial minimal reducing subspaces for M_{B_N} equals the number of connected components of the Riemann surface $B_N^{-1} \circ B_N$ on the unit disk. As verified in [6,7], this result is equivalent to the assertion that $\mathcal{V}^*(B_N)$ is abelian. For infinite Blaschke products, Guo and Huang [8] proved that for "most" thin Blaschke products B, M_B has no nontrivial reducing subspace.

For high-dimensional domains, research on reducing-subspace problems began with some special monomial symbols. Lu and Zhou [14] completely characterized the structure of the reducing subspaces for $M_{z^k w^k}$ on the weighted Bergman spaces over $\mathbb{D}^2$. Shi and Lu [15] found all the minimal reducing subspaces for $M_{z^k w^l} (k \ne l)$ on $A_\alpha^2(\mathbb{D}^n)(\alpha > -1)$ and showed that the unweighted case has more minimal reducing subspaces than the weighted case. Guo and Huang [9] gave the direct decompose of the reducing subspaces for M_{z^a} with $a \in \mathbb{Z}_+^d$ on a multi-dimensional separable Hilbert space by a different approach. For the case that p is a polynomial, the reducing subspaces for $T_{\alpha z^k + \beta w^l} (\alpha, \beta \in \mathbb{C})$ and the structure of $\mathcal{V}^*(\alpha z^k + \beta w^l)$ are investigated in [3,19]. More generally, Guo and Wang [11] studied the reducing subspaces for $A^k \otimes I + I \otimes B^l$ where $A \in B(H)$, $B \in B(K)$ are two simple unilateral weighted shifts.

Motivated by the research of multiplication operators, we wonder what the results about the Toeplitz operator with non-analytic symbols look like. Compared with the analytic conditions, the tools for the Toeplitz operators with general non-analytic symbols seem far fewer at present. Albaseer, Shi and Lu [1] characterized the reducing subspaces for $T_{z^k \overline{w}^l}$ on $A^2(\mathbb{D}^2)$. Let $\varphi(z, w) = \alpha z^k + \beta \overline{w}^l$ where α and β are nonzero complex numbers. In this paper, we find all the minimal reducing subspaces for the Toeplitz operator T_φ on $A^2(\mathbb{D}^2)$, and consider the algebraic structure of $\mathcal{V}^*(\varphi)$. Unlike the analytic condition, we obtain that $\mathcal{V}^*(\varphi)$ is always abelian for every $\alpha\beta \ne 0$. The following theorem is our main result.

Theorem 1.1. *Let $\varphi(z, w) = \alpha z^k + \beta \overline{w}^l$, where α, β are nonzero complex numbers and k, l are positive integers. Then*

$$L_{a,b} = \overline{\text{span}\{z^{a+nk} w^{b+ml} |\ n, m \in \mathbb{Z}_+\}}(0 \le a \le k-1, 0 \le b \le l-1)$$

are exactly all the minimal reducing subspaces for T_φ. Furthermore, $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to

$$\bigoplus_{i=1}^{kl} \mathbb{C},$$

and then $\mathcal{V}^*(\varphi)$ is abelian.