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A proof of Wang–Kooij's conjectures for a cubic Liénard system with a cusp ☆

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ABSTRACT

In this paper the global dynamics of a cubic Liénard system with a cusp is studied to follow Wang and Kooij (1992) [13], who proved that the maximum number of limit cycles is 2 and stated two conjectures about the curves of the cuspidal loop bifurcation and the double limit cycle bifurcation. We give positive answers to those two conjectures and further properties of those bifurcation curves such as monotonicity and smoothness. Finally, associated with previous results we obtain the complete bifurcation diagram and all phase portraits, and demonstrate some numerical examples.

1. Introduction

The planar Liénard system, a representation in the two-dimensional form of the Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$, is one of the classical mechanical systems. The research of its dynamical behaviors can be found in many monographs (see, e.g., [4,8,14]) and many interesting results are given in journal papers (see, e.g., [5,7,10,12,13]). A cubic Liénard system

$$\begin{cases} \dot{x} = y + \mu_1 x^2 + x^3, \\ \dot{y} = \mu_2 x^2 - x^3 \end{cases} \tag{1.1}$$

has been introduced in [1,11,13] to study the viscous flow structures of a three-dimensional system near a planar wall. The origin O is the unique equilibrium when $\mu_2 = 0$. Besides O, system (1.1) has another equilibrium $E : (\mu_2, -\mu_1\mu_2^2 - \mu_2^3)$ and O is a cusp when $\mu_2 \neq 0$. Since the form of (1.1) is invariant under

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the change $(x, y, \mu_1, \mu_2) \to (-x, -y, -\mu_1, -\mu_2)$, we only need to consider (μ_1, μ_2) in $\mathcal{G} := \{(\mu_1, \mu_2) \in \mathbb{R}^2 : \mu_1 \geq 0\}$.

In [13] the global dynamical analysis of system (1.1) is done for $(\mu_1, \mu_2) \in \mathcal{G}$. It is proved that the maximum number of limit cycles is 2. The existence of the cuspidal loop bifurcation curves is given as well as the existence of the double limit cycle bifurcation curves. Moreover, the uniqueness of the cuspidal loop bifurcation curves is also proved and the unique one is denoted by $\mu_1 = \varphi(\mu_2)$. However, there is no answer to the uniqueness of the double limit cycle bifurcation curves. On the other hand, as stated in [13, Theorem 5] $\varphi(\mu_2) \geq \psi_1(\mu_2) := \max\{\mu_1 : (\mu_1, \mu_2)$ lies on the double limit cycle bifurcation curves$\}$ for any fixed μ_2. But we do not know if there exists a point (μ_1, μ_2) lying on both the cuspidal loop bifurcation curve and one of the double limit cycle bifurcation curves, i.e., the location relation of the cuspidal loop bifurcation curve and the double limit cycle bifurcation curves is another unsolved question. Hence, in [13] there are two conjectures:

Conjecture (a) $\varphi(\mu_2) > \psi_1(\mu_2)$.
Conjecture (b) *The double limit cycle bifurcation curve is unique.*

Note that the bifurcation diagram, shown in [13, Figure 5], is given based on that both these conjectures have positive answers. As indicated in the proof of [13, Theorem 5], the stability of the cuspidal loop if it exists is equivalent to $\varphi(\mu_2) > \psi_1(\mu_2)$ because the semistability of the cuspidal loop means $\varphi(\mu_2) = \psi_1(\mu_2)$. Thus, **Conjecture (a)** is actually equivalent to conjecture that the cuspidal loop is stable.

Following the work of [13], we continue to study the global dynamical behaviors of system (1.1). Our main purpose is to answer **Conjectures (a)** and **(b)** so that the bifurcation diagram can be given strictly and to investigate the monotonicity of those bifurcation curves as well as their smoothness. To help the readers and keep the completeness of results, associated with some results of [13, Theorem 5] we give our main result in the following theorem, where large (resp. small) limit cycles mean periodic orbits surrounding two equilibria (resp. a single equilibrium).

Theorem 1.1. *As shown in Fig. 1, the global bifurcation diagram of (1.1) consists of the following bifurcation curves:*

(1) *generalized transcritical bifurcation curve* $GT = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = 0\}$*;*
(2) *Hopf bifurcation curve* $H = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_1 = -3\mu_2/2 > 0\}$ *for* E*;*
(3) *cuspidal loop bifurcation curve* $CL = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_1 = \varphi(\mu_2) > 0\}$*;*
(4) *double limit cycle bifurcation curve* $DL = \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 = \psi(\mu_1) < 0\}$*;*

where $\varphi \in C^\infty(\mathbb{R}^-, \mathbb{R}^+)$ *is decreasing,* $\psi \in C^0(\mathbb{R}^+, \mathbb{R}^-)$ *and*

$$-\mu_2 < \min\{\mu_1 : \mu_2 = \psi(\mu_1)\} \leq \max\{\mu_1 : \mu_2 = \psi(\mu_1)\} < \varphi(\mu_2) < -3\mu_2/2. \tag{1.2}$$

The complete classification of phase portraits is also given in Fig. 1, where

$$I := \{(\mu_1, \mu_2) \in \mathcal{G} : \mu_2 > 0\};$$
$$II := \left\{(\mu_1, \mu_2) \in \mathcal{G} : 0 < \frac{-3\mu_2}{2} < \mu_1\right\};$$
$$III := \left\{(\mu_1, \mu_2) \in \mathcal{G} : \varphi(\mu_2) < \mu_1 < \frac{-3\mu_2}{2}\right\};$$
$$IV := \{(\mu_1, \mu_2) \in \mathcal{G} : \psi(\mu_1) < \mu_2 < \varphi^{-1}(\mu_1)\};$$