J. Math. Anal. Appl. 445 (2017) 944–952

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A note on Mackey topologies on Banach spaces

A.J. Guirao [a,*,1], V. Montesinos [a,2], V. Zizler [b]

[a] Instituto de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera, s/n. 46022 Valencia, Spain
[b] University of Alberta, Edmonton, Alberta, Canada

ARTICLE INFO

Article history:
Received 16 February 2016
Available online 22 August 2016
Submitted by B. Cascales

Keywords:
Banach space
Mackey topology
Completeness
Norming subspace
Mazur space
Weak compactness

ABSTRACT

There is a maybe unexpected connection between three apparently unrelated notions concerning a given w^*-dense subspace Y of the dual X^* of a Banach space X: (i) The norming character of Y, (ii) the fact that (Y, w^*) has the Mazur property, and (iii) the completeness of the Mackey topology $\mu(X, Y)$, i.e., the topology on X of the uniform convergence on the family of all absolutely convex w^*-compact subsets of Y. To clarify these connections is the purpose of this note. The starting point was a question raised by M. Kunze and W. Arendt and the answer provided by J. Bonet and B. Cascales. We fully characterize $\mu(X, Y)$-completeness or its failure in the case of Banach spaces X with a w^*-angelic dual unit ball—in particular, separable Banach spaces or, more generally, weakly compactly generated ones—by using the norming or, alternatively, the Mazur character of Y. We characterize the class of spaces where the original Kunze–Arendt question has always a positive answer. Some other applications are also provided.

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1. Introduction

In [1], J. Bonet and B. Cascales answered in the negative a question of M. Kunze and W. Arendt—see [9]—by showing that in the dual of $\ell_1[0,1]$ there exists a closed and norming—in fact, 1-norming—subspace Y such that $(\ell_1[0,1], \mu(\ell_1[0,1], Y))$ is not complete. Here, $\mu(X, Y)$ stands for the *Mackey topology on X associated to a dual pair* $\langle X, Y\rangle$, i.e., the topology on X of the uniform convergence on the family of all absolutely convex and $w(Y, X)$-compact subsets of Y. In [6] we gave some criteria for deciding whether, for a general Banach space X and a $w(X^*, X)$-dense subspace Y of its dual, the space $(X, \mu(X, Y))$ is—or not—complete. We focussed on subspaces Y that contain a predual of X if available, in contrast with the

* Corresponding author.
E-mail addresses: anguisa2@mat.upv.es (A.J. Guirao), vmontesinos@mat.upv.es (V. Montesinos), vasekzizler@gmail.com (V. Zizler).

1 Supported in part by MICINN and FEDER (Project MTM2014-57838-C2-1-P) and by Fundación Séneca (CARM), Project 19368/PI/14.
2 Supported in part by MICINN (Project MTM2014-57838-C2-2-P).

http://dx.doi.org/10.1016/j.jmaa.2016.08.030
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situation in the original example of Bonet and Cascales. We discussed also the case of the space c_0, that fails to have a predual. It is worth to remind the reader that if X is a Banach space, $\mu(X, X^*)$ is just the topology defined by the norm—and so $(X, \mu(X, X^*))$ is complete—, and that, in the case that X has a predual $P \subset X^*$, the space $(X, \mu(X, P))$ is also complete—a consequence of the Krein–Šmulyan theorem. Even for subspaces Y of X^* such that $P \subset Y$, the completeness of $(X, \mu(X, Y))$ is not guaranteed.

In this note, that extends and completes [1] and [6] (and provides some applications), we observe first that a result on Mazur spaces due to A. Wilansky in [12] gives a unified approach to several situations treated there. The simplicity of the technique used adds extra insight into previous discussions. Then we characterize—by using the concept of norming subspace—the completeness of $(X, \mu(X, Y))$ when the Banach space X has an angelic dual ball, a situation that includes the separable Banach spaces X and, more generally, the weakly compactly generated ones—so in particular the reflexive Banach spaces, although for this last instance our results become irrelevant. Finally, we apply our results, among other things, to characterize those Banach spaces where the answer to the Kunze–Arendt question mentioned above is always positive.

We adopt here the terminology of Banach spaces—as, e.g., in [4]—, even when dealing with locally convex spaces. For example, if X is a locally convex space, then X^* denotes its topological dual. If S is a subset of X^*, the topology on X of the pointwise convergence on points of S will be denoted by $w(X, S)$, and $w(S, X)$ will denote the topology on S of the pointwise convergence on points of X. Quite often, the topology $w(X^*, X)$ will be denoted, as it is usual, by w^*. We shall try to use the same name for a topology on a topological space and its restriction to any subset if no ambiguity is expected. The dual norm of a norm $\|\cdot\|$ will be denoted by $\|\cdot\|^*$, although we shall use $\|\cdot\|$ instead if there is no risk of misunderstanding.

A subspace Y of the dual X^* of a Banach space X is said to be *norming* (1-*norming*) whenever $\|x\|_Y := \sup\{\langle x, y^*\rangle : y^* \in B_Y\}$, $x \in X$, is an equivalent norm (respectively, is the original norm) on X. Observe that every norming subspace of X^* is w^*-dense. If S is a subset of a vector space E, then $[S]$ will denote the linear span of S, and if $x \in E$, then $[x]$ will denote the linear span of the set $\{x\}$. For other non-defined concepts we refer, e.g., to [4]. Our Banach spaces are always assumed to be real.

A. Grothendieck gave a characterization of the completion of a locally convex space—see, e.g., [8, §21.9]—that, when applied to a dual pair $\langle X, Y\rangle$ and the associated Mackey topology $\mu(X, Y)$ on X, reads:

(G) *The completion of the locally convex space $(X, \mu(X, Y))$ can be identified to the set of all linear functionals $L: Y \to \mathbb{R}$ whose restriction $L|_K$ to any absolutely convex and w^*-compact subset K of Y is w^*-continuous.*

In particular, $(X, \mu(X, Y))$ is complete if, and only if, given a linear functional $L: Y \to \mathbb{R}$ whose restriction to any absolutely convex and w^*-compact subset of Y is w^*-continuous, there is $x \in X$ such that $\langle x, y\rangle = L(y)$ for all $y \in Y$—we will simply say that L belongs to X.

2. The main results

A topological space T is said to be *Fréchet–Urysohn* if the sequential closure and the closure of any subset of T agree. It is said to be *angelic*—a concept due to H. D. Fremlin—whenever every relatively countably compact K of T is relatively compact, and the sequential closure of K coincides with its closure. Of course, in the setting of compact topological spaces, both notions coincide. Any Banach space is angelic in its weak topology. If X is a Banach space, the closed unit ball B_{X^*} of its dual X^*, when endowed with the w^*-topology, is angelic if X is *weakly compactly generated*—WCG, in short, meaning that there exists a weakly compact and linearly dense subset of X—in particular, if X is separable. This happens, more generally, if X is weakly Lindelöf determined—see, e.g., [4, Chapter 14].

Along this paper, Y will denote a w^*-dense subspace of the dual X^* of a Banach space X. We shall provide criteria to decide whether $(X, \mu(X, Y))$ is or not complete.

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