J. Math. Anal. Appl. 445 (2017) 985–997

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Entropy numbers of functions on $[-1, 1]$ with Jacobi weights ☆

Kai Wang *, Yali Wang

School of Mathematics and Information Sciences, Langfang Teachers University, Langfang 065000, China

ARTICLE INFO

Article history:
Received 6 May 2016
Available online 23 August 2016
Submitted by S. Tikhonov

Keywords:
Entropy numbers
Weighted Sobolev space
Gaussian quadrature formulae

ABSTRACT

We study the entropy numbers of weighted Sobolev classes $BW^r_{p,\alpha,\beta}$ in $L_{q,\alpha,\beta}$, where $L_{q,\alpha,\beta}, 1 \le q \le \infty$ denotes the weighted L_q space on $[-1, 1]$ with respect to weight $w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta,\ \alpha, \beta > -1/2$. Exact orders of the entropy numbers are obtained for all $1 \le p, q \le \infty$ and $\alpha, \beta > -1/2$.

1. Introduction and main result

Denote by $L_{p,\alpha,\beta} \equiv L_p([-1, 1], w_{\alpha,\beta}),\ 1 \le p < \infty$, the space of measurable functions defined on $[-1, 1]$ with the finite norm

$$\|f\|_{p,\alpha,\beta} := \Big(\int_{-1}^{1} |f(x)|^p\, w_{\alpha,\beta}(x)dx \Big)^{1/p},$$

where $w_{\alpha,\beta}(x) := (1-x)^\alpha(1+x)^\beta,\ \alpha, \beta > -1/2$ is the Jacobi weight. For $p = \infty$ we assume that $L_{\infty,\alpha,\beta}$ is replaced by the space $C[-1, 1]$ of continuous functions on $[-1, 1]$ with the uniform norm.

It is well known that the classical Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ normalized by $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ form an orthogonal basis for $L_{2,\alpha,\beta}$ (see [25]). In particular,

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(y) w_{\alpha,\beta}(x)dx = \delta_{n,m} h_n(\alpha, \beta),$$

☆ Supported by the National Natural Science Foundation of China (Project no. 11271263), the Beijing Natural Science Foundation (1132001), and the Foundation of Langfang Teachers University (LSLB201603).

* Corresponding author.
E-mail addresses: cnuwangk@163.com (K. Wang), 2284690983@qq.com (Y. Wang).

http://dx.doi.org/10.1016/j.jmaa.2016.08.024

where

$$h_n(\alpha,\beta) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} \sim n^{-1}$$

with constants of equivalence depending only on α and β. Then the normalized Jacobi polynomials $P_n(x)$ defined by

$$P_n(x) = (h_n^{(\alpha,\beta)})^{-1/2} P_n^{(\alpha,\beta)}(x), \quad n = 0, 1, \ldots,$$

form an orthonormal basis for $L_{2,\alpha,\beta}$ where the inner product is defined by

$$\langle f, g \rangle := \int_{-1}^{1} f(x)\overline{g(x)} w_{\alpha,\beta}(x)\, dx.$$

Consequently, for every $f \in L_{2,\alpha,\beta}$, $f = \sum_{l=0}^{\infty} \langle f, P_l \rangle P_l$.

We know that $P_n^{(\alpha,\beta)}$ is just the eigenfunction corresponding to the eigenvalues $-n(n+\alpha+\beta+1)$ of the second-order differential operator

$$D_{\alpha,\beta} := (1-x^2)\frac{d^2}{dx^2} - (\alpha - \beta + (\alpha+\beta+2)x)\frac{d}{dx},$$

which means that

$$D_{\alpha,\beta} P_n^{(\alpha,\beta)}(x) = -n(n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(x).$$

Given $r > 0$, we define the fractional power $(-D_{\alpha,\beta})^{r/2}$ of the operator $-D_{\alpha,\beta}$ on f by

$$(-D_{\alpha,\beta})^{r/2}(f) = \sum_{k=0}^{\infty} (k(k+\alpha+\beta+1))^{r/2} \langle f, P_k \rangle P_k,$$

in the sense of distribution.

The weighted Sobolev space is defined as follows: for $r > 0$ and $1 \le p \le \infty$,

$$W_p^r([-1,1], \omega_{\alpha,\beta}) \equiv W_{p,\alpha,\beta}^r := \Big\{ f \in L_{p,\alpha,\beta} : \exists\, g \in L_{p,\alpha,\beta} \text{ such that } g = (-D_{\alpha,\beta})^{\frac{r}{2}}(f) \Big\},$$

where $\|f\|_{W_{p,\alpha,\beta}^r} := \|f\|_{p,\alpha,\beta} + \|(-D_{\alpha,\beta})^{\frac{r}{2}}(f)\|_{p,\alpha,\beta}$. While we denote by $BW_{p,\alpha,\beta}^r$ the unit ball of $W_{p,\alpha,\beta}^r$.

Let A be a compact subset of a Banach space X. For $n \in \mathbb{N}$, the nth entropy number $e_n(A, X)$ is defined as the infimum of all positive ε such that there exist $x_1, \ldots, x_{2^n}$ in X satisfying $A \subset \bigcup_{k=1}^{2^n} (x_k + \varepsilon B_X)$, where B_X is the unit ball of X, that is,

$$e_n(A, X) = \inf\{\varepsilon > 0 : A \subset \bigcup_{k=1}^{2^n} (x_k + \varepsilon B_X),\ x_1, \ldots, x_{2^n} \in X\}.$$

Let $T \in L(X, Y)$ be a bounded linear operator between the Banach spaces X and Y. The nth entropy number $e_n(T)$ is defined as

$$e_n(T) := e_n(T : X \mapsto Y) = e_n(T(B_X), Y).$$