J. Math. Anal. Appl. 418 (2014) 978–984

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Strict inequality of Robin eigenvalues for elliptic differential operators on Lipschitz domains

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ARTICLE INFO

Article history:
Received 20 January 2014
Available online 24 April 2014
Submitted by J.A. Ball

Keywords:
Eigenvalue inequality
Elliptic differential operator
Robin boundary condition

ABSTRACT

On a bounded Lipschitz domain we consider two selfadjoint operator realizations of the same second order elliptic differential expression subject to Robin boundary conditions, where the coefficients in the boundary conditions are functions. We prove that inequality between these functions on the boundary implies strict inequality between the eigenvalues of the two operators, provided that the inequality of the functions in the boundary conditions is strict on an arbitrarily small nonempty, open set.

1. Introduction

We consider an elliptic differential expression of second order of the form

$$\mathcal{L} = - \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k + \sum_{j=1}^{n} (a_j \partial_j - \partial_j \overline{a_j}) + a \tag{1.1}$$

with bounded Lipschitz coefficients on a bounded, connected Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$; see Assumption 2.1 below. Given two real-valued functions $\theta_1, \theta_2 \in L^p(\partial\Omega)$ (for appropriate p, see Assumption 2.2 below) with

$$\theta_1 \leqslant \theta_2 \quad \text{on } \partial\Omega \tag{1.2}$$

we focus on the purely discrete spectra of the selfadjoint operators associated with $\mathcal{L}$ in $L^2(\Omega)$ subject to the Robin boundary conditions

$$\left.\frac{\partial u}{\partial \nu_{\mathcal{L}}}\right|_{\partial\Omega} + \theta_j u|_{\partial\Omega} = 0, \quad j = 1, 2; \tag{1.3}$$

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http://dx.doi.org/10.1016/j.jmaa.2014.04.036

here $u|_{\partial\Omega}$ denotes the trace and $\frac{\partial u}{\partial \nu_{\mathcal{L}}}|_{\partial\Omega}$ is the conormal derivative of u at the boundary $\partial\Omega$; cf. Section 2. The eigenvalues corresponding to (1.3) form a real sequence bounded from below, which accumulates to $+\infty$; we denote these eigenvalues by

$$\lambda_1^{\theta_j} \leqslant \lambda_2^{\theta_j} \leqslant \dots, \quad j = 1, 2,$$

where we count multiplicities. From (1.2) it follows immediately via the variational formulation of the eigenvalue problems that

$$\lambda_k^{\theta_1} \leqslant \lambda_k^{\theta_2}, \quad k \in \mathbb{N}.$$

Our aim in this note is to show that the inequality becomes strict for all k,

$$\lambda_k^{\theta_1} < \lambda_k^{\theta_2}, \quad k \in \mathbb{N},$$

whenever

$$\theta_1|_\omega < \theta_2|_\omega$$

holds for an arbitrary nonempty, open set $\omega \subset \partial\Omega$. This observation complements various results on eigenvalue inequalities for the Laplacians with Dirichlet and Neumann or Dirichlet and Robin boundary conditions, see, e.g., the classical works [10,18,21,22,25] and the more recent contributions [1,9,12,24]. For further investigations of elliptic differential operators subject to (generalized) Robin boundary conditions and of their spectra we refer the reader to [2–4,6,7,11,13,15–17,19,26] and their references.

We wish to remark that if Ω is replaced by the (unbounded) exterior of a bounded Lipschitz domain one can show that the operators corresponding to θ_1 and θ_2 have the same essential spectra. In this case our result remains true for all eigenvalues below the bottom of the joint essential spectrum.

The proof of our result is carried out in Section 3. It adapts Filonov's method in [9] and combines it with a consideration made in [5], based on a unique continuation argument. Before that, in Section 2, we discuss properties of elliptic differential operators with Robin boundary conditions on Lipschitz domains.

2. Elliptic differential operators with Robin boundary conditions on Lipschitz domains

In this section we collect preliminary material on trace maps on Lipschitz domains and recall the definition of selfadjoint elliptic differential operators with Robin boundary conditions via sesquilinear forms.

Let us first fix the assumptions on the domain Ω and the differential expression $\mathcal{L}$.

Assumption 2.1. The set $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, is a bounded, connected Lipschitz domain, see, e.g., [20] for the standard definition. The differential expression $\mathcal{L}$ on Ω is given by (1.1), where $a_{jk}, a_j : \overline{\Omega} \to \mathbb{C}$ are bounded Lipschitz functions satisfying $a_{jk}(x) = \overline{a_{kj}(x)}$ for all $x \in \overline{\Omega}$, and $a : \Omega \to \mathbb{R}$ is measurable and bounded. Moreover, $\mathcal{L}$ is uniformly elliptic on Ω, i.e., there exists $E > 0$ such that

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geqslant E \sum_{k=1}^n \xi_k^2, \quad x \in \overline{\Omega},\ \xi = (\xi_1, \dots, \xi_n)^T \in \mathbb{R}^n.$$

Let us denote by $H^s(\Omega)$ and $H^s(\partial\Omega)$ the Sobolev spaces of orders $s \in \mathbb{R}$ on Ω and its boundary $\partial\Omega$, respectively. Here and in the following $\partial\Omega$ is equipped with the usual surface measure; cf. [20]. Recall that there exists a unique bounded trace map from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$ which extends the mapping