J. Math. Anal. Appl. 419 (2014) 606–616

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the existence of non-monotone non-oscillating wavefronts

Anatoli Ivanov [a], Carlos Gomez [b], Sergei Trofimchuk [b,*]

[a] *Department of Mathematics, Pennsylvania State University, P.O. Box PSU, Lehman, PA 18627, USA*
[b] *Instituto de Matemática y Fisica, Universidad de Talca, Casilla 747, Talca, Chile*

ARTICLE INFO

Article history:
Received 28 October 2013
Available online 9 May 2014
Submitted by R. Popovych

Keywords:
Monostable nonlinearity
Diffusive Mackey–Glass equation
Delay
Wavefront
Non-monotone response

ABSTRACT

We present a monostable delayed reaction–diffusion equation with the unimodal birth function which admits only non-monotone wavefronts. Moreover, these fronts are either eventually monotone (in particular, such is the minimal wave) or slowly oscillating. Hence, for the Mackey–Glass type diffusive equations, we answer affirmatively the question about the existence of non-monotone non-oscillating wavefronts. As it was recently established by Hasik et al. and Ducrot et al., the same question has a negative answer for the KPP-Fisher equation with a single delay.

1. Introduction and main results

This work deals with the traveling waves for the diffusive Mackey–Glass type equation

$$u_t(t,\mathbf{x}) = \Delta u(t,\mathbf{x}) - u(t,\mathbf{x}) + g\big(u(t-h,\mathbf{x})\big), \quad u(t,\mathbf{x}) \geqslant 0, \ \mathbf{x} \in \mathbb{R}^m. \tag{1}$$

The population model (1) was extensively studied (including its non-local version) during the past decade, e.g. see [10,11,16–18,24] and references therein. Notice that the non-negativity condition $u(t,\mathbf{x}) \geqslant 0$ of (1) is due to the biological interpretation of u as the size of an adult population. In this paper we are mostly concerned with classical positive solutions to (1) of the special form $u(t,\mathbf{x}) = \phi(ct + \mathbf{n}\cdot\mathbf{x})$, $c > 0$, $|\mathbf{n}| = 1$, where ϕ additionally satisfies the boundary conditions $\phi(-\infty) = 0$, $\phi(+\infty) = \kappa$. Such solutions of Eq. (1) are called traveling fronts or simply wavefronts. The function $\phi : \mathbb{R} \to \mathbb{R}_+$ is said to be the profile of the wavefront $u(t,\mathbf{x}) = \phi(ct + \mathbf{n}\cdot\mathbf{x})$. It is easy to see that each profile ϕ is a positive heteroclinic solution of the delay differential equation

$$x''(t) - cx'(t) - x(t) + g\big(x(t-ch)\big) = 0, \quad t \in \mathbb{R}. \tag{2}$$

* Corresponding author.
E-mail addresses: afi1@psu.edu (A. Ivanov), cgomez@inst-mat.utalca.cl (C. Gomez), trofimch@inst-mat.utalca.cl (S. Trofimchuk).

http://dx.doi.org/10.1016/j.jmaa.2014.04.075
0022-247X/

The nonlinear term g in (1) and (2) plays the role of a *birth function* and therefore it is non-negative. Motivated by various concrete applications, throughout the paper we assume that g satisfies the following unimodality condition (see also Fig. 2):

(UM) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and has only one positive local extremum point $x = \theta$ (global maximum). Furthermore, g has two equilibria $g(0) = 0$, $g(\kappa) = \kappa$ with $g'(0) > 1$, $g'(\kappa) < 1$ and additionally satisfies $g(x) > x$ for $x \in (0, \kappa)$ and $g(x) < x$ for $x > \kappa$.

Therefore, in view of the terminology used in the traveling waves theory, the diffusive Mackey–Glass type equation (1) is of monostable type [11]. In the particular case when g is monotone on the interval $[0, \kappa]$ there is quite satisfactory description of all wavefront solutions for Eq. (1) given by the following result.

Proposition 1. *(See [15,22].) Suppose that g satisfies* **(UM)** *and is strictly monotone on $[0, \kappa]$, see Fig. 2 (left). Then there is $c_* > 0$ (called the minimal speed of propagation) such that Eq. (1) has a unique (up to translation) wavefront $u(t, \mathbf{x}) = \phi(ct + \mathbf{n} \cdot \mathbf{x})$ for each $c \geqslant c_*$ and every $h \geqslant 0$. In addition, the profile ϕ is a strictly increasing function. If $c < c_*$ then Eq. (1) does not have any wavefront.*

We note that the stability of monotone fronts of (1) was successfully analyzed in [16–18].

In the case when $\theta \in (0, \kappa)$ (so that g is not monotone on $[0, \kappa]$ anymore), much less information on the traveling fronts to Eq. (1) is available. In particular, as far as we know, for a general function g satisfying the hypothesis **(UM)**, none of the four aspects (the existence of the minimal speed c_*, the uniqueness, the monotonicity properties, the wavefront stability) mentioned in Proposition 1 has received a satisfactory characterization. In this paper, we shed some new light on the description of possible geometric shapes of the wavefront profiles ϕ. Due to the biological interpretation of solutions to (1), the geometric properties of leading (invading) parts of wavefront profiles characterize the 'smoothness' of the expansion (invasion) processes. This fact shows the practical importance of our studies. A first picture of the wavefront monotonicity properties was obtained in [24] under the following additional condition:

(FC) The restriction $g : [g^2(\theta), g(\theta)] \to \mathbb{R}_+$ has the positive feedback with respect to the equilibrium κ: $(g(x) - \kappa)(x - \kappa) < 0$, $x \neq \kappa$. Here we use the notation $g^2(\theta)$ for $g(g(\theta))$.

More precisely, the following result holds:

Proposition 2. *(See [24].) Consider the case when* **(UM)** *holds and $g'(\kappa) < 0$. Let $u(t, \mathbf{x}) = \phi(ct + \mathbf{n} \cdot \mathbf{x})$ be a wavefront to Eq. (1). Then there exists $\tau_1 \in \mathbb{R} \cup \{+\infty\}$ such that $\phi'(s) > 0$ on $(-\infty, \tau_1)$. Furthermore, τ_1 is finite if and only if $\phi(\tau_1) > \kappa$. If, in addition, the birth function g satisfies* **(FC)***, then ϕ is eventually either monotone or slowly oscillating around κ. Finally, if τ_0 is the leftmost point where $\phi(\tau_0) = \theta$ then $\tau_1 - \tau_0 \geqslant ch$.*

It should be noted that the existence of oscillating traveling fronts in the delayed reaction–diffusion equations is by now a well-known fact confirmed both numerically and analytically. The subclass of slowly oscillating profiles is defined below:

Definition 3. Set $\mathbb{K} = [-ch, 0] \cup \{1\}$. For any $v \in C(\mathbb{K}) \setminus \{0\}$ we define the number of sign changes by

$$\mathrm{sc}(v) = \sup\bigl\{k \geqslant 1 : \text{there are } t_0 < \cdots < t_k \text{ such that } v(t_{i-1})v(t_i) < 0 \text{ for } i \geqslant 1\bigr\}.$$