J. Math. Anal. Appl. 417 (2014) 537–551

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Gradient regularity for solutions to quasilinear elliptic equations in the plane

Linda Maria De Cave[a,*], Carlo Sbordone[b]

[a] Sapienza Università di Roma, Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Via A. Scarpa 16, 00161 Roma, Italy
[b] Università degli studi di Napoli Federico II, Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Via Cintia, 80126 Napoli, Italy

ARTICLE INFO

Article history:
Received 8 November 2013
Available online 19 March 2014
Submitted by P. Koskela

Keywords:
Elliptic equations
Zygmund spaces
Gradient regularity

ABSTRACT

We investigate the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(x,\nabla v) = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

for a quasilinear elliptic equation in a planar domain Ω, when f belongs to the Zygmund space $L(\log L)^{\frac{1}{2}}(\log\log L)^{\epsilon}(\Omega)$, $0<\epsilon<1$. We prove that the gradient of the variational solution $v \in W_0^{1,2}(\Omega)$ belongs to the space $L^2(\log\log L)^{2\epsilon}(\Omega;\mathbb{R}^2)$. A main tool is a result on the regularity of the gradient of the solution φ to the Dirichlet problem

$$\begin{cases} \operatorname{div} a(x,\nabla\varphi) = \operatorname{div} \underline{\chi} & \text{in } \Omega \\ \varphi \in W_0^{1,1}(\Omega) \end{cases}$$

where $\underline{\chi} \in L^2(\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)$, $\beta>0$. Namely, if the mapping $a:\Omega\times\mathbb{R}^2\to\mathbb{R}^2$ satisfies the Leray–Lions type conditions, then we prove the estimates

$$\|\nabla\varphi\|_{L^2(\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)} \leqslant C(\beta)\|\underline{\chi}\|_{L^2(\log\log L)^{-\beta}(\Omega;\mathbb{R}^2)}$$

by applying a method recently suggested by L. Greco et al., which is based on the uniform estimates

$$\|\nabla\varphi\|_{L^{2-\sigma}(\Omega;\mathbb{R}^2)} \leqslant C\|\underline{\chi}\|_{L^{2-\sigma}(\Omega;\mathbb{R}^2)}$$

available for $|\sigma| \leqslant \sigma_0$ provided that $\underline{\chi} \in L^{2-\sigma}(\Omega;\mathbb{R}^2)$.

* Corresponding author.
E-mail addresses: linda.decave@sbai.uniroma1.it (L.M. De Cave), sbordone@unina.it (C. Sbordone).

http://dx.doi.org/10.1016/j.jmaa.2014.03.035
0022-247X/

1. Introduction

Let $\Omega \subset \mathbb{R}^{\mathrm{N}}$ ($\mathrm{N} \geqslant 2$) be a bounded open set with C^1 boundary.

The Dirichlet problem for elliptic equations of Leray–Lions type

$$\begin{cases} -\operatorname{div} a(x, \nabla v) = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

with $f \in L^1(\Omega)$, has been studied by various authors in the past.

For example, in the papers by G. Stampacchia [25], by L. Boccardo and T. Gallouët [5], by F. Murat [23] and by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [10], it has been proved the existence of different kinds of weak solutions v, respectively duality solutions, distributional solutions, transposition solutions and renormalized solutions, in the linear case or in the nonlinear case, but all of these such that $v \in \bigcap_{q \in [1, \frac{\mathrm{N}}{\mathrm{N}-1})} W_0^{1,q}(\Omega)$.

Instead, in [14], A. Fiorenza and C. Sbordone studied the planar case, using the *grand Sobolev space* $W_0^{1,2)}(\Omega)$ as a natural space both for existence and uniqueness of distributional solutions (see also [17]). We underline that the space $W_0^{1,2)}(\Omega)$ is slightly larger than the classical Sobolev space $W_0^{1,2}(\Omega)$ and slightly smaller than $\bigcap_{q \in [1,2)} W_0^{1,q}(\Omega)$.

We assume that the function $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a mapping of Leray–Lions, that is, for almost every $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^2$, the ellipticity and growth conditions

$$\bigl|a(x,\xi) - a(x,\eta)\bigr| \leqslant K|\xi - \eta| \tag{2}$$

$$|\xi - \eta|^2 \leqslant K\bigl\langle a(x,\xi) - a(x,\eta), \xi - \eta\bigr\rangle \tag{3}$$

$$a(x,0) = 0 \tag{4}$$

with $K \geqslant 1$ hold true.

The case which is critical with respect to *finite energy* solutions, i.e. the solutions v which satisfy $v \in W_0^{1,2}(\Omega)$, is the one in which the right hand side f belongs to the Orlicz space $L(\log L)^{\frac{1}{2}}(\Omega)$. This is a consequence of the Sobolev–Trudinger imbedding in the plane

$$W_0^{1,2}(\Omega) \hookrightarrow EXP_2(\Omega),$$

that, since every function in $W_0^{1,2}(\Omega)$ can be approximated in norm with an $L^\infty(\Omega)$ sequence (the sequence of the truncates), implies

$$W_0^{1,2}(\Omega) \hookrightarrow \exp_2(\Omega)$$

from which, by duality, it follows

$$L(\log L)^{\frac{1}{2}}(\Omega) \hookrightarrow W^{-1,2}(\Omega)$$

(see Section 2 for the definitions of all these functional spaces). This condition guarantees that, if $f \in L(\log L)^{\frac{1}{2}}(\Omega)$, then the solution v of (1) enjoys the regularity $v \in W_0^{1,2}(\Omega)$ (see [22]).

Further regularity derives from the stronger assumption $f \in L\log L(\Omega)$. Actually in [3,2,6,7] the following proprieties were proved for $\mathrm{N} = 2$

$$f \in L\log L(\Omega) \quad \Longrightarrow \quad |\nabla v| \in L^2 \log L(\Omega),\ v \in \exp(\Omega),\ v \in C^0(\Omega).$$