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Original article

Method of corrections by higher order differences for Poisson equation with nonlocal boundary conditions

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Abstract

We consider the Bitsadze–Samarskii type nonlocal boundary value problem for Poisson equation in a unit square, which is solved by a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme converges at the rate $O(|h|^s)$ in the discrete L_2-norm provided that the solution of the original problem belongs to the Sobolev space with exponent $s \in [2, 4]$.

Keywords: Nonlocal BVP; Difference scheme; Method of corrections; Improvement of accuracy; Compatible estimates of convergence rate

1. Introduction

Finite difference method is a significant tool in the numerical solution of problems posed for differential equations. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse meshes, i.e. to have high order accuracy. In the present work, for improving the accuracy of the approximate solution, we study two-stage finite difference method. We consider Bitsadze–Samarskii type nonlocal boundary value problem for Poisson's equation.

At the first stage we solve the difference scheme $\Delta_h \tilde{U} = \varphi$, which has the second order of approximation. Using the solution $\tilde{U}$ the right-hand side of the difference scheme is corrected, $\Delta_h U = \varphi + R\tilde{U}$, and solved again on the same mesh.

This approach for some boundary value problems posed for Poisson and Laplace equations has been studied in Volkov's papers (see, e.g. [1–3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$.

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For establishing the convergence we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4–6]), and later in the works of other authors (see, e.g., [7,8]). For the elliptic problems such estimates have the form

$$\|U-u\|_{W_2^k(\omega)} \le c|h|^{s-k}\|u\|_{W_2^s(\Omega)}, \quad s>k\ge 0,$$

where u is the solution of original problem, U is the approximate solution, k and s are integer and real numbers, respectively, $W_2^k(\omega)$ and $W_2^s(\Omega)$ are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below c denotes a positive generic constant, independent of h and u.

It is proved that the solution U of the corrected difference scheme converges at rate $O(h^s)$ in the discrete L_2-norm, when the exact solution belongs to the Sobolev space W_2^s, $s\in[2,4]$.

The generalization of the Bitsadze–Samarskii problem [9] was investigated by many authors (see, e.g., [10–13]).

In [11] for a Poisson equation it is considered a difference scheme, which converges by the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar{\Omega})$.

In [13] difference scheme is considered for a second order elliptic equation with variable coefficients and the compatible estimate of convergence rate in discrete W_2^1-norm is obtained.

Results, analogous to those given in the present work, are obtained in [14] for the Dirichlet problem posed for an elliptic equation, and also in [15] for the mixed problem with third kind conditions.

One of the methods for obtaining compact high order approximations is the Mehrstellen method (“Mehrstellenverfahren”), defined by Collatz (see [16]). Instead of approximating only the left hand side of the differential equation, he proposes to take several points of the right hand side as well. In the case of two-dimensional problem, the differential operator is approximated on a 9-point stencil with the fourth order accuracy.

The advantage of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid is obvious.

The advantage of our method is:

(a) It needs to approximate the differential operator on minimally acceptable stencil (5-point stencil for a two-dimensional problem). Therefore, the condition number of this operator is better as compared with the Mehrstellen schemes, which is notable on a fine grid.

(b) It is a two-stage method, nevertheless it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

(c) The method of correction is handy even in the case when construction of high precision schemes is impossible.

2. Statement of the problem and some auxiliary estimate

As usual, by symbol $W_2^s(\Omega)$, $s\ge 0$ we denote the Sobolev space. For integer s the norm in $W_2^s(\Omega)$ is given by formula

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s} |u|^2_{W_2^j(\Omega)}, \qquad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j} \|D^\nu u\|^2_{L_2(\Omega)},$$

where $D^\nu = \partial^{|\nu|}/\left(\partial x_1^{\nu_1}\partial x_2^{\nu_2}\right)$, $\nu=(\nu_1,\nu_2)$ is multi-index with non-negative integer components, $|\nu|=\nu_1+\nu_2$.

If $s=\bar{s}+\varepsilon$, where $\bar{s}$ is an integer part of s and $0<\varepsilon<1$, then

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\bar{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|_{W_2^s(\Omega)} = \sum_{|\nu|=\bar{s}} \int_\Omega\int_\Omega \frac{|D^\nu u(x)-D^\nu u(y)|^2}{|x-y|^{2+2\varepsilon}}\,dx\,dy.$$

Particularly, for $s=0$ we have $W_2^0=L_2$.