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On γ-positive polynomials arising in Pattern Avoidance

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ABSTRACT

A classical result of Foata and Schützenberger states that the γ-coefficients of the Eulerian polynomials enumerate permutations without double descents by the number of descents. In this paper, based on works of Cheng et al. and Stankova, we provide similar combinatorial interpretations for the γ-coefficients of the inversion polynomials on 321-avoiding permutations and the descent polynomials on Separable permutations (or $(2413, 3142)$-avoiding permutations) and $(1342, 2431)$-avoiding permutations. Some further open problems are also proposed.

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1. Introduction

Inspired by recent work of Cheng et al. [7] on 321-avoiding permutations and a positivity conjecture of Gessel (see [3, Conjecture 10.2] and [13]) on double Eulerian polynomials, we investigate in this paper several γ-positive polynomials arising in Pattern Avoidance. Let us first review some partial and related results on this topic.

A polynomial $h(t) = \sum_{i=0}^{d} h_i t^i$ with palindromic (or symmetric) coefficients ($h_{d-i} = h_i$) can be expanded uniquely as $h(t) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k t^k (1+t)^{d-2k}$, and is said to be *$\gamma$-positive* if $\gamma_k \geq 0$ for all k. It is clear that γ-positivity implies palindromicity and unimodality, since each term $t^k(1+t)^{d-2k}$ is palindromic (with the same center $d/2$) and unimodal.

Let $\mathfrak{S}_n$ denote the set of all permutations of $[n] := \{1, 2, \dots, n\}$. For each permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n$ in $\mathfrak{S}_n$, an index $i \in [n-1]$ is a *decent* (resp. *double descent*) of σ if $\sigma_i > \sigma_{i+1}$ (resp. $i \geq 2$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$). Denote by $\mathrm{des}(\sigma)$ the number of descents of σ. The descent polynomial $A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)}$, also known as the *$n$-th Eulerian polynomial*, has been demonstrated to be γ-positive. Moreover, if we introduce

$$\mathrm{NDD}_n := \{\sigma \in \mathfrak{S}_n : \sigma \text{ has no double descents}\}, \quad \text{then}$$

Theorem 1.1 *(Foata–Schützenberger [11]). For $n \geq 1$,*

$$A_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k}, \tag{1.1}$$

where $\gamma_{n,k} = |\{\sigma \in \mathrm{NDD}_n : \sigma_1 < \sigma_2, \mathrm{des}(\sigma) = k\}|$.

Many proofs for this classical result (and its various generalizations) are known: via recurrence relations [16], the *modified Foata–Strehl action* (*MFS-action* for short) [3,17,19], *cd*-index [23, Section 1.6.3], continued fractions [20] or quasi-symmetric functions [26], to name a few. Undoubtedly, the most elegant proof is by using the MFS-action, an approach which can also be applied immediately to the descent polynomials over 231-avoiding permutations in (1.2) below.

Recall that a permutation σ is said to *avoid the permutation (or pattern)* π if there does not exist a subsequence of (not necessarily consecutive) entries of σ which is order isomorphism to π. For $S \subseteq \mathfrak{S}_n$, the set of permutations in S avoiding patterns $\pi_1, \dots, \pi_r$ is denoted by $S(\pi_1, \dots, \pi_r)$. A result attributed to MacMahon and Knuth is that, for each $\pi \in \mathfrak{S}_3$, $\mathfrak{S}_n(\pi)$ is counted by the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. The reader is referred to the recent survey of Vatter [25] for more information about Pattern Avoidance.

The *Narayana polynomials* $N_n(t)$ are defined by

$$N_n(t) := \sum_{k=0}^{n-1} \frac{1}{n} \binom{n}{k+1} \binom{n}{k} t^k,$$