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Random cyclic dynamical systems ☆

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For X a finite subset of the circle and for $0 < r \leq 1$ fixed, consider the function $f_r \colon X \to X$ which maps each point to the clockwise furthest element of X within angular distance less than $2\pi r$. We study the discrete dynamical system on X generated by f_r, and especially its expected behavior when X is a large random set. We show that, as $|X| \to \infty$, the expected fraction of periodic points of f_r tends to 0 if r is irrational and to $\frac{1}{q}$ if $r = \frac{p}{q}$ is rational with p and q coprime. These results are obtained via more refined statistics of f_r which we compute explicitly in terms of (generalized) Catalan numbers. The motivation for studying f_r comes from Vietoris–Rips complexes, a geometric construction used in computational topology. Our results determine how much one can expect to simplify the Vietoris–Rips complex of a random sample of the circle by removing dominated vertices.

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1. Introduction

Finite cyclic dynamical systems We are interested in a family of finite dynamical systems parametrized by a finite subset $X \subseteq S^1$ of the circle and a real number $0 < r \leq 1$. Here $S^1 = \mathbb{R}/\mathbb{Z}$ is the circle of unit circumference equipped with the arc-length metric. For $X \subseteq S^1$ finite and $0 < r \leq 1$, we define the map $f_r\colon X \to X$ which sends each point $x \in X$ to the furthest element of X within clockwise distance less than r from x (equivalently, within angular distance less than $2\pi r$ from x). Iterating f_r gives rise to a discrete time dynamical system on X, which we call a *cyclic* dynamical system. We can speak of *periodic* and *non-periodic* points of f_r and the structure and size of the *periodic orbits* of f_r (see Fig. 1(a,b)).

As the set $X \subseteq S^1$ becomes more dense in S^1, each cyclic dynamical system f_r can be seen as a discrete approximation of the rigid rotation of the circle by angle $2\pi r$.

Random cyclic systems — main results To get a sample of the circle one can take a random set $X = X_n$ of n points chosen uniformly and independently from S^1. What is the asymptotic behavior of the cyclic dynamical systems $f_r : X_n \to X_n$ as $n \to \infty$? Our main results analyze the number of periodic points (denoted $\mathrm{per}(X, r)$) and the structure of periodic orbits.

Main Theorem. *Let X_n be a sample of n points chosen uniformly and independently from S^1 and let $0 < r \leq 1$.*

- *The expected fraction of periodic points in $f_r : X_n \to X_n$ is*

$$\lim_{n\to\infty} \frac{\mathbf{E}[\mathrm{per}(X_n, r)]}{n} = \begin{cases} 0 & \text{if } r \text{ is irrational,} \\ \frac{1}{q} & \text{if } r = \frac{p}{q} \text{ is rational.}^{1} \end{cases}$$

- *If r is rational, then asymptotically almost surely there is one periodic orbit.*
- *If r has irrationality exponent 2, then the expected number of periodic points satisfies $\mathbf{E}[\mathrm{per}(X_n, r)] = \Omega(n^{1/2-\varepsilon})$ for any $\varepsilon > 0$.*

The main theorem is a combination of Theorems 4.6, 4.7, and 5.3.

Our proofs rely on a more refined count of the non-periodic points of dynamical system f_r. We say a non-periodic point $x \in X$ is at *level* $i \geq 0$ if $x \in f_r^i(X) \setminus f_r^{i+1}(X)$; let $\mathrm{lev}_i(X, r)$ denote the number of non-periodic points at level i. Let C_i be the i-th Catalan number, i.e. the number of Dyck paths from $(0, 0)$ to $(2i, 0)$, and let $C_{i,q-2}$ be the number of Dyck paths of height at most $q - 2$.

[1] Throughout the paper, whenever r is written as $r = \frac{p}{q}$ it is understood that $p, q \in \mathbb{Z}$ are relatively prime.