

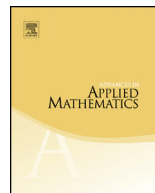


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Advances in Applied Mathematics

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Random cyclic dynamical systems<sup>☆</sup>Michał Adamaszek<sup>a</sup>, Henry Adams<sup>b,\*</sup>, Francis Motta<sup>c</sup><sup>a</sup> Department of Mathematical Sciences, University of Copenhagen, Denmark<sup>b</sup> Department of Mathematics, Colorado State University, USA<sup>c</sup> Department of Mathematics, Duke University, USA

## ARTICLE INFO

*Article history:*

Received 4 July 2016

Received in revised form 28 August 2016

Accepted 30 August 2016

Available online xxxx

*MSC:*

60C05

37H99

05E45

*Keywords:*

Discrete dynamical systems

Geometric probability

Catalan numbers

Vietoris–Rips complexes

## ABSTRACT

For  $X$  a finite subset of the circle and for  $0 < r \leq 1$  fixed, consider the function  $f_r: X \rightarrow X$  which maps each point to the clockwise furthest element of  $X$  within angular distance less than  $2\pi r$ . We study the discrete dynamical system on  $X$  generated by  $f_r$ , and especially its expected behavior when  $X$  is a large random set. We show that, as  $|X| \rightarrow \infty$ , the expected fraction of periodic points of  $f_r$  tends to 0 if  $r$  is irrational and to  $\frac{1}{q}$  if  $r = \frac{p}{q}$  is rational with  $p$  and  $q$  coprime. These results are obtained via more refined statistics of  $f_r$  which we compute explicitly in terms of (generalized) Catalan numbers. The motivation for studying  $f_r$  comes from Vietoris–Rips complexes, a geometric construction used in computational topology. Our results determine how much one can expect to simplify the Vietoris–Rips complex of a random sample of the circle by removing dominated vertices.

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<sup>☆</sup> MA supported by Villum Fonden through the network for Experimental Mathematics in Number Theory, Operator Algebras, and Topology. HA supported in part by Duke University and by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation. The research of MA and HA was supported through the program “Research in Pairs” by the Mathematisches Forschungsinstitut Oberwolfach in 2015. FM supported by research training grant award number NSF-RTG DMS-10-45153 to John Harer with funds provided by the National Science Foundation, and grants N66001-15-2-4073 and DARPA-D12AP00025 to John Harer, Duke University, with funds provided by the Defense Advanced Research Projects Agency.

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## 1. Introduction

*Finite cyclic dynamical systems* We are interested in a family of finite dynamical systems parametrized by a finite subset  $X \subseteq S^1$  of the circle and a real number  $0 < r \leq 1$ . Here  $S^1 = \mathbb{R}/\mathbb{Z}$  is the circle of unit circumference equipped with the arc-length metric. For  $X \subseteq S^1$  finite and  $0 < r \leq 1$ , we define the map  $f_r: X \rightarrow X$  which sends each point  $x \in X$  to the furthest element of  $X$  within clockwise distance less than  $r$  from  $x$  (equivalently, within angular distance less than  $2\pi r$  from  $x$ ). Iterating  $f_r$  gives rise to a discrete time dynamical system on  $X$ , which we call a *cyclic* dynamical system. We can speak of *periodic* and *non-periodic* points of  $f_r$  and the structure and size of the *periodic orbits* of  $f_r$  (see Fig. 1(a,b)).

As the set  $X \subseteq S^1$  becomes more dense in  $S^1$ , each cyclic dynamical system  $f_r$  can be seen as a discrete approximation of the rigid rotation of the circle by angle  $2\pi r$ .

*Random cyclic systems — main results* To get a sample of the circle one can take a random set  $X = X_n$  of  $n$  points chosen uniformly and independently from  $S^1$ . What is the asymptotic behavior of the cyclic dynamical systems  $f_r: X_n \rightarrow X_n$  as  $n \rightarrow \infty$ ? Our main results analyze the number of periodic points (denoted  $\text{per}(X, r)$ ) and the structure of periodic orbits.

**Main Theorem.** *Let  $X_n$  be a sample of  $n$  points chosen uniformly and independently from  $S^1$  and let  $0 < r \leq 1$ .*

- *The expected fraction of periodic points in  $f_r: X_n \rightarrow X_n$  is*

$$\lim_{n \rightarrow \infty} \frac{\mathbf{E}[\text{per}(X_n, r)]}{n} = \begin{cases} 0 & \text{if } r \text{ is irrational,} \\ \frac{1}{q} & \text{if } r = \frac{p}{q} \text{ is rational.} \end{cases}^1$$

- *If  $r$  is rational, then asymptotically almost surely there is one periodic orbit.*
- *If  $r$  has irrationality exponent 2, then the expected number of periodic points satisfies  $\mathbf{E}[\text{per}(X_n, r)] = \Omega(n^{1/2-\varepsilon})$  for any  $\varepsilon > 0$ .*

The main theorem is a combination of [Theorems 4.6, 4.7, and 5.3](#).

Our proofs rely on a more refined count of the non-periodic points of dynamical system  $f_r$ . We say a non-periodic point  $x \in X$  is at *level*  $i \geq 0$  if  $x \in f_r^i(X) \setminus f_r^{i+1}(X)$ ; let  $\text{lev}_i(X, r)$  denote the number of non-periodic points at level  $i$ . Let  $C_i$  be the  $i$ -th Catalan number, i.e. the number of Dyck paths from  $(0, 0)$  to  $(2i, 0)$ , and let  $C_{i, q-2}$  be the number of Dyck paths of height at most  $q - 2$ .

<sup>1</sup> Throughout the paper, whenever  $r$  is written as  $r = \frac{p}{q}$  it is understood that  $p, q \in \mathbb{Z}$  are relatively prime.

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