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Permutations sortable by two stacks in series

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ABSTRACT

We address the problem of the number of permutations that can be sorted by two stacks in series. We do this by first counting all such permutations of length less than 20 exactly, then using a numerical technique to obtain nineteen further coefficients approximately. Analysing these coefficients by a variety of methods we conclude that the OGF behaves as

$$S(z) \sim A(1-\mu \cdot z)^{\gamma},$$

where $\mu = 12.45 \pm 0.15$, $\gamma = 1.5 \pm 0.3$, and $A \approx 0.02$.

1. Introduction

In the late 1960s Knuth [8] introduced the idea of classifying the common data structures of computer science in terms of the number of permutations of length n that could be sorted by the given data structure, to produce the identity permutation. Knuth demonstrated the usefulness of this approach by showing that a simple stack could sort all such permutations except those which had any three elements in relative order 231. This restriction meant that of the $n!$ possible permutations of length n, only $C_n \sim 4^n/(n^{3/2}\sqrt{\pi})$ could be sorted by a simple stack. Here C_n denotes the nth Catalan number. Knuth went on to pose the same question for more complex data structures,

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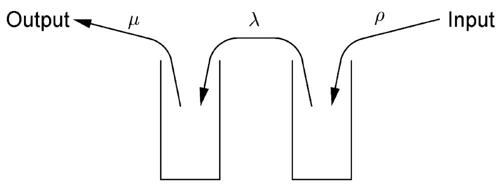

Fig. 1. Two stacks in series.

such as a double-ended queue or *deque*, which is a linear list in which insertions and deletions can take place at either end. In a later volume of his celebrated book [9], he asked the same question about compositions of stacks.

The three most interesting, and most intensively studied permutation-related sorting problems associated with data structures relate to permutations that can be sorted by (i) a deque, (ii) two stacks in parallel (2SIP) and (iii) two stacks in series (2SIS). The data structure corresponding to two stacks in series is shown in Fig. 1. A permutation of length n is said to be *sortable* if it is possible to start with this permutation as the input, and output the numbers $1, 2, \ldots, n$ in order, using only the moves ρ, λ and μ in some order. Here ρ pushes the next element from the input onto the first stack, λ pushes the top element of the first stack onto the top of the second stack, and μ outputs (pops) the top element of the second stack to the output stream, as shown in Fig. 1.

Recently Albert and Bousquet-Mélou [2] solved the problem relating to two stacks in parallel, while subsequently we [5] related the solution of the 2SIP problem to the solution of the deque problem. This leaves only the 2SIS problem unresolved. Significant progress has been made on subsets of that problem. For example Atkinson, Murphy and Ruškuc [3] solved the problem in the case of *sorted* stacks, while Elder, Lee and Rechnitzer [4] solved the problem in the case when one of the stacks is of depth 2. Unfortunately, both these cases correspond to an exponentially small subset of the full set of stack-sortable permutations. In [12], Pierrot and Rossin give a polynomial algorithm to decide if a given permutation is sortable by two stacks in series.

In all cases we've mentioned, the number of permutations of length n that can be sorted by the given data structure grows exponentially (just as in the simple stack case discussed above). Indeed, the Stanley–Wilf conjecture, subsequently proved by Marcus and Tardos [10], states that for every permutation π, there is a constant μ such that the number of permutations of length n which avoid π as a permutation pattern is at most μ^n. Additionally, it is expected, but not proved, that p_n, the number of permutations of length n sortable by any of the afore-mentioned data structures, behaves as $p_n \sim const \cdot \mu^n \cdot n^g$ in general. The dominant exponential growth term is a consequence of the Marcus–Tardos theorem, but the sub-dominant term n^g is conjectural. In [1], rigorous upper and lower bounds on μ are given for deque sorting, and also for 2SIP and 2SIS. For 2SIS the bounds are $8.156 < \mu < 13.374$.

In this paper we give an alternative approximation. We have evaluated the exact number of stack-sortable permutations of length n for $n < 20$, and describe numerical techniques that give the approximate number for $20 \leq n \leq 38$. We then apply a range