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Perturbation bounds of generalized inverses☆

Lingsheng Meng[a], Bing Zheng[a,*], Peilan Ma[b]

[a] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China
[b] Department of Mathematics, Yili Normal University, Yining 835000, PR China

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ABSTRACT

Let complex matrices A and B have the same sizes. We characterize the generalized inverse matrix $B^{(1,\,i)}$, called an $\{1, i\}$-inverse of B for each $i = 3$ and 4, such that the distance between a given $\{1, i\}$-inverse of a matrix A and the set of all $\{1, i\}$-inverses of the matrix B reaches minimum under 2-norm (spectral norm) and Frobenius norm. Similar problems are also studied for $\{1, 2, i\}$-inverse. In practice, the matrix B is often considered as the perturbed matrix of A, and hence based on the previous results, the additive perturbation bounds for the $\{1, i\}$- and $\{1, 2, i\}$-inverses and multiplicative perturbation bounds for the $\{1\}$-, $\{1, i\}$- and $\{1, 2, i\}$-inverses are proposed. Numerical examples show that these multiplicative perturbation bounds can be achieved respective under 2-norm and Frobenius norm.

1. Introduction

Let $C^{m\times n}$ and $C_r^{m\times n}$ denote the set of all $m \times n$ complex matrices and the subset of $C^{m\times n}$ consisting of all matrices with rank r, respectively. We use A^*, $\|A\|_2$ and $\|A\|_F$ respectively stand for the conjugate transpose, the spectral norm and the Frobenius norm of a matrix A, and $\|A\|_{2,F}$ when either can be used (consistently throughout an expression). As usual, I_m means the identity matrix of order m. Moreover, we also denote the following orthogonal projections,

$$P_A = AA^\dagger, \quad P_A^\perp = I_m - AA^\dagger, \quad P_{A^*} = A^\dagger A, \quad P_{A^*}^\perp = I_n - A^\dagger A,$$

where $A \in C^{m\times n}$ and $A\dagger$ denotes the Moore–Penrose inverse of A.

Recall that a generalized inverse $G \in C^{n\times m}$ of $A \in C^{m\times n}$ is a matrix which satisfies some of the following four Moore–Penrose equations [1]:

$$(1)\ AGA = A, \quad (2)\ GAG = G, \quad (3)\ (AG)^* = AG, \quad (4)\ (GA)^* = GA.$$

Let $\{i, j, k\} \subseteq \{1, 2, 3, 4\}$. Then $A\{i, j, k\}$ denotes the set of all matrices G which satisfy equations (i), (j) and (k). Any $G \in A\{i, j, k\}$ is called an $\{i, j, k\}$-inverse of A. For examples, a matrix G of the set $A\{1\}$ is called a $\{1\}$-inverse of A and denoted by $A^{(1)}$. Any matrix G in the set $A\{1, 3\}$ is called a $\{1, 3\}$-inverse of matrix A and denoted by $A^{(1,\,3)}$, which is also called a least squares g-inverse of A since any solution x of the least square problem $\min_{x\in C^n} \|Ax - b\|_2$ can be represented as $x = A^{(1,3)}b$. A $\{1, 4\}$-inverse of A is denoted by $A^{(1,\,4)}$, which is also called a minimum norm g-inverse of A as any minimum norm solution x of the consistent linear equation $Ax = b$ can be expressed as $x = A^{(1,4)}b$. Similarly, any $\{1, 2, k\}$-inverse of A is denoted by

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* Corresponding author.
E-mail address: bzheng@lzu.edu.cn (B. Zheng).

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$A^{(1,\,2,\,k)}$, $k=3,4$. The unique {1, 2, 3, 4}-inverse of A is denoted by $A\dagger$, which is called the Moore–Penrose inverse of A. We refer the readers to [1] for basic results on generalized inverses.

Due to the uncertainty or inaccuracy of the data in practical applications such as statistics, numerical computations and etc., the perturbation theory of generalized inverses has received considerable attention during the past decades, see [1,8–16] and their references therein. However, in these works, most attention was given to the additive perturbation model, i.e., A is perturbed to $B=A+E$, see e.g., [8–16]. As a special additive perturbation model, the multiplicative perturbation model, i.e., $A\in C^{m\times n}$ is perturbed to $B=D_1^*AD_2$, is often encountered in the areas such as numerical analysis and statistics etc., and has been considered by some authors, for example, see [3,5–8,10], where D_1 and D_2 are respectively $m\times m$ and $n\times n$ nonsingular matrices. However, it seems that the perturbation analysis for the nearest {1}-, {1, i}- and {1, 2, i}-inverses with respect to the multiplicative perturbation model has not been discussed yet in the literature. In addition, most of the existing works were done when the generalized inverse is unique (for example, the Moore–Penrose inverse, group inverse), see, e.g., [8,10,12–14,16]. For the case that the generalized inverse is not unique, Liu et al. [9] recently studied the continuity properties of {1}-inverses under condition of rank invariant perturbations, and Wei and Ling [15] studied the perturbation bounds of {1}-inverses under the additive perturbation model. As the continuous works, in this paper we further undertake the perturbation analysis for {1, i}- and {1, 2, i}-inverses of a matrix, $i=3,4$. Different from the approaches in [9,15], our discussions are mainly based on the SVD method. Beside the additive perturbation model, the multiplicative perturbation model is also included in our consideration.

The rest of this paper is organized as follows. In Section 2, we give some lemmas which will be used in the later discussions. In Section 3, for any given $A^{(1,\,i)}\in A\{1,\,i\}$ $(i=3,4)$ and matrix $B\in C^{m\times n}$, we first find a matrix $B_m^{(1,i)}\in B\{1,i\}$ such that $B_m^{(1,i)}$ is the closest matrix to $A^{(1,\,i)}$ under 2-norm and Frobenius norm using the SVD method; and then similar problems are discussed for any given $A^{(1,\,2,\,i)}\in A\{1,\,2,\,i\}$. The general expressions of the {1, 2, i}-inverses $B_m^{(1,2,i)}\in B\{1,2,i\}$ such that $B_m^{(1,2,i)}$ is the closest matrix to $A^{(1,\,2,\,i)}$ under the Frobenius norm and 2-norm are derived, where $i=3,4$. Using the results in Section 3, Section 4 gives the additive perturbation bounds for the nearest perturbed {1, i}- and {1, 2, i}-inverses. The multiplicative perturbation bounds for {1}-, {1, i}- and {1, 2, i}-inverses are described in Section 5. Some numerical examples are provided to show that all of the multiplicative perturbation bounds under 2-norm and Frobenius norm are optimal.

2. Lemmas

In this section, we present some lemmas which will be used in the following sections.

Lemma 2.1 [1]. Let $A\in C_r^{m\times n}$, *then the general expressions of* {1}-, {1, 3}- *and* {1, 2, 3}-*inverses of A can be written as:*

$$A\{1\}=\{A^\dagger+A^\dagger AZ(I_m-AA^\dagger)+(I-A^\dagger A)Z:Z\in C^{n\times m}\},$$
$$A\{1,3\}=\{A^\dagger+(I-A^\dagger A)Z:Z\in C^{n\times m}\}$$

and

$$A\{1,2,3\}=\{A^\dagger+(I_n-A^\dagger A)ZAA^\dagger:Z\in C^{n\times m}\}.$$

Furthermore, let $A=U\begin{pmatrix}\Sigma_1 & 0\\ 0 & 0\end{pmatrix}V^*$ *be the singular value decomposition (SVD) of A, where* $U\in C^{m\times m}$, $V\in C^{n\times n}$ *are unitary matrices, and* $\Sigma_1=diag(\sigma_1,\dots,\sigma_r)$. *Then the general* {1}-*inverse of A can be expressed as*

$$A^{(1)}=V\begin{pmatrix}\Sigma_1^{-1} & K\\ L & M\end{pmatrix}U^*, \tag{2.1}$$

where K, L, M are arbitrary submatrices of appropriate sizes. In particular, $K=0$ *gives the general* {1, 3}-*inverse;* $K=0,\ M=0$ *gives the general* {1, 2, 3}-*inverse; and the Moore–Penrose inverse is* (2.1) *with* $K=0,\ L=0$ *and* $M=0$.

Lemma 2.2 [13]. *Let* $A,B=A+E\in C^{m\times n}$, *then*

$$B^\dagger-A^\dagger=-B^\dagger EA^\dagger+B^\dagger(I_m-AA^\dagger)-(I_n-B^\dagger B)A^\dagger. \tag{2.2}$$

Next lemma, which was originally studied by Davis et al. [4] and recently slightly generalized by Wei and Ling [15], is very useful in the norm-preserving dilation and optimal error estimates.

Lemma 2.3 [4,15]. *Given* $A\in C^{m\times n}$, $B\in C^{p\times n}$, $C\in C^{m\times q}$ *such that*

$$\left\|\begin{pmatrix}A\\ B\end{pmatrix}\right\|_2=\mu_1\quad\text{and}\quad\|(A,C)\|_2=\mu_2,$$

and $\mu=\max\{\mu_1,\mu_2\}$, *then there exists* $D\in C^{p\times q}$ *such that*

$$\min_{D\in C^{p\times q}}\left\|\begin{pmatrix}A & C\\ B & D\end{pmatrix}\right\|_2=\mu.$$