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A parameter uniform difference scheme for singularly perturbed parabolic delay differential equation with Robin type boundary condition

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ABSTRACT

A Robin type boundary value problem for a singularly perturbed parabolic delay differential equation is studied on a rectangular domain in the x - t plane. The second-order space derivative is multiplied by a small parameter, which gives rise to parabolic boundary layers on the two lateral sides of the rectangle. A numerical method comprising a standard finite difference scheme on a rectangular piecewise uniform fitted mesh of $N_x \times N_t$ elements condensing in the boundary layers is suggested and it is proved to be parameter-uniform. More specifically, it is shown that the errors are bounded in the maximum norm by $C(N_x^{-2} \ln^2 N_x + N_t^{-1})$, where C is a constant independent of N_x, N_t and the small parameter. To validate the theoretical result an example is provided.

1. Introduction

Singularly perturbed delay partial differential equations provide more realistic models for phenomena in many areas of science (such as population dynamics) that display time-lag or after-effect than do conventional instantaneous singularly perturbed non delay partial differential equations.

Singularly perturbed non delay partial differential equations relate an unknown function to its derivatives evaluated at the same instance. In contrast, singularly perturbed delay partial differential equations model physical problems for which the evolution does not only depend on the present state of the system but also on the past history. Singularly perturbed non delay partial differential equations have been studied extensively by many authors (see [1–12] and the references therein). However, the basic theory and numerical methods of singularly perturbed delay partial differential equations are still at the initial stage.

Delay partial differential equations arise from many biological, chemical and physical systems which are characterised by both spatial and temporal variables and exhibit various spatio-temporal patterns. Examples occur in population ecology (to describe the interaction of spatial diffusion and time delays), generic repression (taking into account time delays from processes of transcription and translation as well as spatial diffusion of reactants in the models), and modelling size-dependent cell growth and division. Examples for delay partial differential equations from numerical control and population dynamics are given in [13].

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Boundary layers occur in the solution of singularly perturbed problems when the singular perturbation parameter, which multiplies terms involving the highest derivatives in the differential equation, tends to zero. These boundary layers are located in neighbourhoods of the boundary of the domain, or sometimes interior (discontinuous source term), where the solution has a very steep gradient. Away from any corner of the domain a boundary layer of either regular or parabolic type may occur. A boundary layer is said to be parabolic if the characteristics of the reduced equation (for $\varepsilon = 0$) are parallel to the boundary and regular if these characteristics are not parallel to the boundary. In the vicinity of a corner, a boundary layer is said to be of corner type [5].

Ansari et al. [13] proposed a parameter uniform finite difference scheme for singularly perturbed delay parabolic partial differential equations of reaction diffusion type. They made use of the results of Miller et al. [14]. Kaushik et al. [15] considered convection diffusion type equations. Bashier and Patidar [16] improved the order of accuracy for the method suggested in [13]. A second-order fitted operator finite difference method for reaction diffusion type equation was suggested by Bashier and Patidar [17]. Kumar and Kumar [18] gave a higher order and also parameter uniform scheme.

The above authors considered differential equations subject to Dirichlet type boundary conditions. Motivated by these papers and [19], we, in this paper, present a parameter uniform finite difference method for singularly perturbed parabolic delay differential equation subject to Robin type boundary condition.

In Section 2 the problem under study is formulated. Also maximum principle and stability result are given. Bounds on the solution and its derivatives are derived in Section 3. A finite difference scheme is proposed in Section 4. In the same section, we present discrete maximum principle and corresponding stability result. Section 5 deals with convergence of the numerical method. An example is given in Section 6 to illustrate the theory. The paper concludes with a discussion.

Throughout our analysis C is a generic positive constant that is independent of parameter ε, number of mesh points N_x and N_t. Further, the supremum norm is used for studying the convergence of the numerical solution to the exact solution of a singular perturbation problem and denoted by $\|\cdot\|_S$ where S is an appropriate domain.

2. Statement of the problem

Let $\Omega = (0, 1)$, $D = \Omega \times (0, T]$ and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where $\Gamma_l = \{0\} \times (0, T]$, $\Gamma_r = \{1\} \times (0, T]$ and $\Gamma_b = [0, 1] \times [-\tau, 0]$. The problem under study is the following singularly perturbed parabolic delay differential equation with Robin type boundary condition:

$$\begin{cases} L_\varepsilon u_\varepsilon(x,t) \equiv \left(\frac{\partial u_\varepsilon}{\partial t} - \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2} + a u_\varepsilon\right)(x,t) = -b(x,t)u_\varepsilon(x,t-\tau) + f(x,t), \quad (x,t) \in D,\\ B_{l,\varepsilon}u_\varepsilon(x,t) \equiv \left(u_\varepsilon - \sqrt{\varepsilon}\frac{\partial u_\varepsilon}{\partial x}\right)(x,t) = \phi_l(t), \quad (x,t) \in \Gamma_l,\\ B_{r,\varepsilon}u_\varepsilon(x,t) \equiv \left(u_\varepsilon + \sqrt{\varepsilon}\frac{\partial u_\varepsilon}{\partial x}\right)(x,t) = \phi_r(t), \quad (x,t) \in \Gamma_r,\\ u_\varepsilon(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b, \end{cases} \tag{1}$$

where $0 < \varepsilon \le 1$ and $\tau > 0$ (note that $T = k\tau$ for some integer $k > 1$) are given constant, $a(x, t)$, $b(x, t)$, $f(x, t)$, $\phi_l(t)$, $\phi_r(t)$ and $\phi_b(x, t)$ are sufficiently smooth and bounded functions that satisfy

$$a(x,t) \ge \alpha > 0, \quad b(x,t) \ge \beta > 0, \quad (x,t) \in \overline{D}.$$

The reduced problem corresponding to (1) is

$$\begin{cases} \frac{\partial u_0}{\partial t}(x,t) + a(x,t)u_0(x,t) = -b(x,t)u_0(x,t-\tau) + f(x,t), \quad (x,t) \in D,\\ u_0(x,t) = \phi_b(x,t), \quad (x,t) \in \Gamma_b. \end{cases} \tag{2}$$

Then it is obvious that the solution of (1) has boundary layers on Γ_l and Γ_r. The characteristics of (2) are the vertical lines $x =$ constant, which implies that boundary layers arising in the solution are of parabolic type.

The problem (1) satisfies the following continuous maximum principle.

Theorem 1. *Assume that $a \in C^{(0,0)}(\overline{D})$ and let $\psi \in U^* = C^{(2,1)}(D) \cap C^{(1,0)}(D^*) \cap C^{(0,0)}(\overline{D})$ be any function satisfying $L_\varepsilon\psi(x, t) \ge 0$, $(x, t) \in D$, $B_{l,\varepsilon}\psi(x, t) \ge 0$, $(x,t) \in \Gamma_l$, $B_{r,\varepsilon}\psi(x, t) \ge 0$, $(x,t) \in \Gamma_r$ and $\psi(x, t) \ge 0$, $(x,t) \in \Gamma_0$, where $D^* = D \cup \Gamma_l \cup \Gamma_r$, $\Gamma_0 = \{(x,0) : x \in [0,1]\}$. Then $\psi(x, t) \ge 0$, $(x,t) \in \overline{D}$.*

Proof. Assume that the function ψ takes its minimum value at a point (x^*, t^*) and this minimum is negative, i.e. $\psi(x^*,t^*) = \min_{(x,t)\in\overline{D}} \psi(x,t) < 0$. Clearly $(x^*,t^*) \notin \Gamma_0$.

Case (i). $(x^*, t^*) \in D$

We have

$$\frac{\partial\psi}{\partial t}(x^*,t^*) \le 0 \quad \text{and} \quad \frac{\partial^2\psi}{\partial x^2}(x^*,t^*) \ge 0.$$

Hence

$$L_\varepsilon\psi(x^*,t^*) = \frac{\partial\psi}{\partial t}(x^*,t^*) - \varepsilon\frac{\partial^2\psi}{\partial x^2}(x^*,t^*) + a\psi(x^*,t^*) < 0,$$

which is a contradiction.