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On weighted regularity criteria for the axisymmetric Navier–Stokes equations☆

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ABSTRACT

In this paper, we consider the axisymmetric Navier–Stokes equations with swirl. By invoking the magic identity of Miao and Zheng, the symmetry properties of Riesz transforms and the Hardy–Sobolev inequality, we establish regularity criterion involving $r^d\omega^\theta$ with $-1 \leq d < 0$. This improves and extends the results of [3,21].

1. Introduction

The three-dimensional Navier–Stokes equations governing the incompressible fluid read as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \triangle u + \nabla\pi = 0, \\ \nabla\cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $u = (u^1, u^2, u^3)$ is the fluid velocity field, π is a scalar pressure, and $\boldsymbol{u}_0$ is the prescribed initial data satisfying the compatibility condition $\nabla\cdot u_0 = 0$ in the sense of distributions.

For initial data of finite energy, a class of weak solutions (the so-called Leray–Hopf weak solutions) satisfying

$$u \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$$

was constructed by Leray [12] and Hopf [6]. However, the issue of regularity and uniqueness of $\boldsymbol{u}$ is an outstanding open problem in mathematical fluid dynamics. Thus, many researchers are devoted to finding sufficient conditions (called the regularity criterion) to ensure the smoothness of the solution after the works of [17,18].

In this paper, we study the axisymmetric solutions to (1.1) of the form

$$u = u^r(t,r,z)e_r + u^z(t,r,z)e_\theta + u^z(t,r,z)e_z, \tag{1.2}$$

where

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right) = (\cos\theta, \sin\theta, 0),$$
$$e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right) = (-\sin\theta, \cos\theta, 0),$$

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$$e_z = (0, 0, 1),$$

and u^r, u^θ and u^z are called the radial, swirl and axial components of $\boldsymbol{u}$ respectively. Thus, (1.1) can be reformulated as

$$\begin{cases} \frac{\tilde{D}}{Dt}u^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^r - \frac{(u^\theta)^2}{r} + \partial_r \pi = 0, \\ \frac{\tilde{D}}{Dt}u^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)u^\theta + \frac{u^r u^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}u^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)u^z + \partial_z \pi = 0, \\ \partial_r(ru^r) + \partial_z(ru^z) = 0, \\ (u^r, u^\theta, u^z)(0) = (u_0^r, u_0^\theta, u_0^z), \end{cases} \tag{1.3}$$

where

$$\frac{\tilde{D}}{Dt} = \partial_t + u^r\partial_r + u_z\partial_z \tag{1.4}$$

denotes the convection derivative (or material derivative). We can also compute the vorticity $\omega = \nabla \times u$ as

$$\omega = \nabla \times u = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z \tag{1.5}$$

with

$$\omega^r = -\partial_z u^\theta, \quad \omega^\theta = \partial_z u^r - \partial_r u^z, \quad \omega^z = \partial_r u^\theta + \frac{u^\theta}{r}. \tag{1.6}$$

The governing equations of $\boldsymbol{\omega}$ can be easily deduced from (1.3) as

$$\begin{cases} \frac{\tilde{D}}{Dt}\omega^r - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^r - (\omega^r\partial_r + \omega^z\partial_z)u^r = 0, \\ \frac{\tilde{D}}{Dt}\omega^\theta - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\right)\omega^\theta - \frac{2u^\theta\partial_z u^\theta}{r} - \frac{u^r\omega^\theta}{r} = 0, \\ \frac{\tilde{D}}{Dt}\omega^z - \left(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\right)\omega^z - (\omega^r\partial_r + \omega^z\partial_z)u^z = 0. \end{cases} \tag{1.7}$$

In case $u^\theta = 0$, we have known the global regularity of (1.3), see [10,13,19]. However, when $u^\theta \neq 0$, the global strong solutions to (1.3) is still unknown. There are many sufficient conditions to ensure the regularity of the solution, see [2–5,7–9,11,15,16,20,21] for example. In particular, we have the following regularity criterion

(1)

$$r^d u^r \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \quad \frac{3}{1-d} < \beta \leq \infty \tag{1.8}$$

with $d = 0$ provided in [15] and the extension to $-1 \leq d < 1$ in [21];

(2)

$$r^d u^\theta \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \quad \frac{3}{1-d} < \beta \leq \infty \tag{1.9}$$

with $0 \leq d < 1$ treated in [3, Theorem 1.1] and $-1 \leq d < 0$ (under the assumption $ru_0^\theta \in L^\infty(\mathbb{R}^3)$) covered by interpolation method in [3, Remark 1.3];

(3)

$$r^d u^z \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 1 - d, \quad \frac{3}{1-d} < \beta \leq \infty, \ 0 \leq d < 1 \tag{1.10}$$

established in [3, Theorem 1.4];

(4)

$$r^d \omega^z \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 - d, \quad \frac{3}{2-d} < \beta < \infty \tag{1.11}$$

with $d = 0$ obtained in [3, Theorem 1.3] and the improvement to $-2 \leq d < 2$ in [21];

(5)

$$r^d \omega^\theta \in L^\alpha(0, T; L^\beta(\mathbb{R}^3)), \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2 - d, \quad \frac{3}{2-d} < \beta < \infty \tag{1.12}$$

with $d = 0$ found in [2, Theorem 1] and the refinement to $0 \leq d < 2$ in [21].