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Oscillation of functional trinomial differential equations with positive and negative term☆

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ABSTRACT

In the paper, we establish new technique for investigation of properties of trinomial differential equations with positive and negative term

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0.$$

We offer new criteria for asymptotic properties of nonoscillatory solutions for the studied equations. We support our results with illustrative examples.

1. Introduction

We shall study asymptotic properties of the third order trinomial differential equation with positive and negative term

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) - q(t)h(x(\sigma(t))) = 0. \tag{E}$$

Throughout the paper, it is assumed that the following hypothesis hold true

(H_1) $a(t)$, $b(t)$, $p(t)$, $q(t)$, $\tau(t)$, $\sigma(t) \in C([t_0, \infty))$ are positive;
(H_2) $\int_{t_0}^{\infty} \frac{1}{b(s)}\,ds = \int_{t_0}^{\infty} \frac{1}{a(s)}\,ds = \infty$;
(H_3) $\sigma(t) \ge t$, $\sigma(t)$ is nondecreasing, $\lim_{t\to\infty} \tau(t) = \infty$;
(H_4) $f(u), h(u) \in C(\mathbb{R})$, $uf(u) > 0$, $uh(u) > 0$ for $u \ne 0$, f is bounded, h is nondecreasing.

By a solution of (E) we mean a function $x(t)$ with $b(t)(a(t)x'(t))' \in C^1([T_x, \infty))$, $T_x \ge t_0$, which satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros and otherwise, it is called nonoscillatory. Eq. (E) is said to be oscillatory if all its solutions are oscillatory.

Setting either $p(t) \equiv 0$ or $q(t) \equiv 0$ Eq. (E) reduces to simpler binomial differential equations

$$\left(b(t)\left(a(t)x'(t)\right)'\right)' + p(t)f(x(\tau(t))) = 0, \tag{E_f}$$

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and

$$\Big(b(t)\big(a(t)x'(t)\big)'\Big)' - q(t)h(x(\sigma(t))) = 0. \tag{E_h}$$

Properties of both equations have been studied by many authors. See papers of Baculikova and Džurina [5,6], Candan and Dahiya [10], Grace et al. [14], Thandapany and Li [20], Tiryaki and Atkas [21].

The well known lemma of Kiguradze [15] implies that the solutions' spaces of (E_f) and (E_h) are absolutely different. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of considered equations, then for (E_f) the set $\mathcal{N}$ has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2,$$

where positive solution

$$x(t) \in \mathcal{N}_0 \iff a(t)x'(t) < 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' < 0$$

$$x(t) \in \mathcal{N}_2 \iff a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' < 0.$$

While, for (E_h) the set $\mathcal{N}$ has the following reduction

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3,$$

with positive solution

$$x(t) \in \mathcal{N}_1 \iff a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' < 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' > 0$$

$$x(t) \in \mathcal{N}_3 \iff a(t)x'(t) > 0, \quad b(t)\big(a(t)x'(t)\big)' > 0, \quad \Big(b(t)\big(a(t)x'(t)\big)'\Big)' > 0.$$

It is understandable that in generally the nonoscillatory solutions' space of (E) with positive and negative part is unclear.

In this paper, we offer new method that overcome those difficulties caused by presence of negative and positive terms of (E). Throughout the paper, we assume that

$$(H_5)\ \int_{t_0}^{\infty} \frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(s)} \int_s^{\infty} p(u)\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t < \infty,$$

and this assumption implies that the negative term is dominating and the structure of the nonoscillatory solutions of (E) is similar that of (E_h).

2. Main results

In this paper, we reduce investigation of trinomial equation onto the oscillation of the suitable first order differential equation. We establish new comparison method for investigating properties of trinomial differential equation with positive and negative term.

Theorem 1. *Assume that for every $k > 0$*

$$\int_{t_0}^{\infty} q(s) \left| h\left(\pm k \int_{t_0}^{\sigma(s)} \frac{1}{a(u)} \int_{t_0}^{u} \frac{1}{b(v)}\,\mathrm{d}v\,\mathrm{d}u \right) \right| \mathrm{d}s = \infty. \tag{2.1}$$

If the first order advanced differential equation

$$y'(t) - \left[\frac{1}{a(t)} \int_t^{\infty} \frac{1}{b(u)} \int_u^{\infty} q(s)\,\mathrm{d}s\,\mathrm{d}u \right] h(y(\sigma(t))) = 0 \tag{E_0}$$

is oscillatory, then every nonoscillatory solution $x(t)$ of (E) satisfies either

$$\lim_{t\to\infty} \frac{|x(t)|}{\int_{t_1}^{t} \frac{1}{a(s)} \int_{t_1}^{s} \frac{1}{b(u)}\,\mathrm{d}u\,\mathrm{d}s} = \infty. \tag{2.2}$$

or

$$\lim_{t\to\infty} x(t) = 0 \quad \text{with}$$
$$|x(t)| \le k \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} \frac{1}{b(u)} \int_u^{\infty} p(v)\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s, \qquad k > 0. \tag{2.3}$$

Proof. Assume that (E) possesses a nonoscillatory solution $x(t)$. Without loss of generality we may assume that $x(t)$ is eventually positive. We introduce the auxiliary function

$$w(t) = x(t) - \int_t^{\infty} \frac{1}{a(s)} \int_s^{\infty} \frac{1}{b(u)} \int_u^{\infty} p(v) f(x(\tau(v)))\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s. \tag{2.4}$$