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The generalized 3-connectivity of graph products☆

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ABSTRACT

The generalized k-connectivity $\kappa_k(G)$ of a graph G, which was introduced by Chartrand et al. (1984), is a generalization of the concept of vertex connectivity. For this generalization, the generalized 2-connectivity $\kappa_2(G)$ of a graph G is exactly the connectivity $\kappa(G)$ of G. In this paper, let G be a connected graph of order n and let H be a 2-connected graph. For Cartesian product, we show that $\kappa_3(G\square H) \geq \kappa_3(G)+1$ if $\kappa(G)=\kappa_3(G)$; $\kappa_3(G\square H) \geq \kappa_3(G)+2$ if $\kappa(G) > \kappa_3(G)$. Moreover, above bounds are sharp. As an example, we show that $\kappa_3(\overbrace{C_{n_1}\square C_{n_2}\square \cdots C_{n_k}}^{k}) = 2k-1$, where C_{n_i} is a cycle. For lexicographic product, we prove that $\kappa_3(H\circ G) \geq \max\{3\delta(G)+1, \lceil\frac{3n+1}{2}\rceil\}$ if $\delta(G) < \frac{2n-1}{3}$, and $\kappa_3(H\circ G)=2n$ if $\delta(G) \geq \frac{2n-1}{3}$.

1. Introduction

All graphs in this paper are undirected, finite and simple. We refer to the book [1] for graph theoretic notations and terminology not described here. The generalized connectivity of a graph G, which was introduced by Chartrand et al. [2], is a natural and nice generalization of the concept of vertex connectivity.

A tree T is called an *S-tree* (*$\{u_1, u_2, \ldots, u_k\}$-tree*) if $S \subseteq V(T)$, where $S=\{u_1, u_2, \ldots, u_k\} \in V(G)$. A family of trees $T_1, T_2, \ldots, T_r$ are *internally disjoint S-trees* if $E(T_i)\cap E(T_j)=\emptyset$ and $V(T_i)\cap V(T_j)=S$ for any pair of integers i and j, where $1 \leq i < j \leq r$. We use $\kappa(S)$ to denote the greatest number of internally disjoint S-trees. For an integer k with $2 \leq k \leq n$, the generalized *k-connectivity* $\kappa_k(G)$ of G is defined as $\min\{\kappa(S) \,|\, S \in V(G)$ and $|S|=k\}$. Clearly, when $|S|=2$, $\kappa_2(G)$ is nothing new but the connectivity $\kappa(G)$ of G, that is, $\kappa_2(G)=\kappa(G)$, which is the reason why one addresses $\kappa_k(G)$ as the generalized connectivity of G. By convention, for a connected graph G with less than k vertices, we set $\kappa_k(G)=1$. For any graph G, clearly, $\kappa(G) \geq 1$ if and only if $\kappa_3(G) \geq 1$.

In addition to being a natural combinatorial measure, the generalized connectivity can be motivated by its interesting interpretation in practice. For example, suppose that G represents a network. If one considers to connect a pair of vertices of G, then a path is used to connect them. However, if one wants to connect a set S of vertices of G with $|S| \geq 3$, then a tree has to be used to connect them. This kind of tree with minimum order for connecting a set of vertices is usually called a Steiner tree, and popularly used in the physical design of VLSI, see [17]. Usually, one wants to consider how tough a network can be, for the connection of a set of vertices. Then, the number of totally independent ways to connect them is a measure for this purpose. The generalized k-connectivity can serve for measuring the capability of a network G to connect any k vertices in G.

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Determining $\kappa_k(G)$ for general graph is non-trivial problem. In [7], Li and Li derived that for any fixed integer $k \geq 2$, given a graph G and a subset S of $V(G)$, deciding whether there are k internally disjoint trees connecting S, namely deciding whether $\kappa(S) \geq k$, is NP-complete. The exact values of $\kappa_k(G)$ are known for only a small class of graphs, examples are complete graphs [3], complete bipartite graphs [6], complete equipartition 3-partite graphs [9], star graphs and bubble-sort graphs [10], Cayley graphs generated by trees and cycles [11] connected Cayley graphs on Abelian groups with small degrees [18]. Upper bounds and lower bounds of generalized connectivity of a graph have been investigated by Li et al. [4,8,14] and Li and Mao [15]. Extremal problem have been investigated by Li et al. [12,13]. We refer the readers to [16] for more results.

In [3], Chartrandet al. determined generalized k-connectivity of complete graphs.

Theorem 1 [3]**.** *For every two integers n and k with $2 \leq k \leq n$, $\kappa_k(K_n) = n - \lceil \frac{k}{2} \rceil$.*

In [8], Li et al. showed the following upper bound of generalized 3-connectivity of a graph.

Theorem 2 [8]**.** *Let G be a connected graph with at least three vertices. If G has two adjacent vertices with minimum degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Theorem 3 [8]**.** *Let G be a connected graph with n vertices. Then $\kappa_3(G) \leq \kappa(G)$. Moreover, the upper bound is sharp.*

In [5], Li et al. obtained the following upper bound of k-connectivity of a graph.

Theorem 4 [5]**.** *For any graph G with order at least k,*

$$\kappa_k(G) \leq \min_{S \subseteq V(G), |S|=k} \left\lfloor \frac{1}{k-1} |E(G[S])| + \frac{1}{k} |E[S, \bar{S}]| \right\rfloor,$$

where $S \subseteq V(G)$ with $|S| = k$, and $\bar{S} = V(G) \setminus S$. Moreover, the bound is sharp.

In [4], Li et al. studied the generalized 3-connectivity of Cartesian product graphs and showed the following result.

Theorem 5 [4]**.** *Let G and H be connected graphs such that $\kappa_3(G) \geq \kappa_3(H)$. The following assertions hold:*

(i) *If $\kappa(G) = \kappa_3(G)$, then $\kappa_3(G \Box H) \geq \kappa_3(G) + \kappa_3(H) - 1$. Moreover, the bound is sharp;*
(ii) *If $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G \Box H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

Note that above bounds are not sharp for some graph class, for example, cycles. For two cycles C_1 and C_2 with length 3, we know that $\kappa(C_1) = \kappa_3(C_1) = 1$ and $\kappa(C_2) = \kappa_3(C_2) = 1$. Thus $\kappa_3(C_1 \Box C_2) \geq 1$ by Theorem 5. In fact, it is not difficult to check that $\kappa_3(C_1 \Box C_2) = 3$. Therefore, it is meaningful to investigate the generalized 3-connectivity of the Cartesian product of a graph and a 2-connected graph.

In [15], Li and Mao studied the generalized 3-connectivity of the lexicographic product graphs and obtained the following result.

Theorem 6 [15]**.** *Let G and H be two connected graphs. Then $\kappa_3(G \circ H) \geq \kappa_3(G)|V(G)|$. Moreover, the bound is sharp.*

It is easy to check that this result is also not sharp for the lexicographic product of two cycles with length 3. Therefore, it is also meaningful to investigate the generalized 3-connectivity of the lexicographic product of a graph and a 2-connected graph.

This paper is organized as follows. In Section 2, we introduce some definitions and notations. In Section 3, we investigate the generalized 3-connectivity of the Cartesian product of a graph and a 2-connected graph. In Section 4, we investigate the generalized 3-connectivity of the lexicographic product of a graph and a 2-connected graph.

2. Definitions and notations

We use P_n to denote a path with n vertices. A path P is called a *u-v path*, denoted by P_{uv}, if u and v are the endpoints of P. Let C be a cycle with vertex set $V(C) = \{v_0, v_1, \ldots, v_{n-1}\}$. Two vertices v_i and v_j are adjacent if and only if $|i - j| = 1 \pmod n$. We use $P_{v_i v_j}$ and $P_{v_j v_i}$ to denote the path $v_i v_{i+1} \ldots v_j$ and $v_j v_{j+1} \ldots v_i$, respectively.

Recall that the *Cartesian product* (also called the *square product*) of two graphs G and H, written as $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \Box H \cong H \Box G$.

Let G and H be two graphs with $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$, respectively. We use $G(u_j, v_i)$ to denote the subgraph of $G \Box H$ induced by the set $\{(u_j, v_i) \mid 1 \leq j \leq n\}$. Similarly, we use $H(u_j, v_i)$ to denote the subgraph of $G \Box H$ induced by the set $\{(u_j, v_i) \mid 1 \leq i \leq m\}$. It is easy to see $G(u_{j_1}, v_i) = G(u_{j_2}, v_i)$ for different u_{j_1} and u_{j_2} of G. Thus, we can replace $G(u_j, v_i)$ by G^{v_i} for simplicity. Similarly, we can replace $H(u_j, v_i)$ by H^{u_j}. The following serval mappings are particularly useful for our proofs.

Given a vertex $v_a \in V(H)$, define

$u^{v_a} := (u, v_a)$ for any vertex $u \in V(G)$,

$G_1^{v_a} := (V(G_1^{v_a}), E(G_1^{v_a}))$ for any subgraph $G_1 \subseteq G$,