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Edge disjoint paths in hypercubes and folded hypercubes with conditional faults

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ABSTRACT

It is known that edge disjoint paths is closely related to the edge connectivity and the multicommodity flow problems. In this paper, we study the edge disjoint paths in hypercubes and folded hypercubes with edge faults. We first introduce the F-strongly Menger edge connectivity of a graph, and we show that in all n-dimensional hypercubes (folded hypercubes, respectively) with at most $2n-4(2n-2,$ respectively) edges removed, if each vertex has at least two fault-free adjacent vertices, then every pair of vertices u and v are connected by $min\{deg(u), deg(v)\}$ edge disjoint paths, where $deg(u)$ and $deg(v)$ are the remaining degree of vertices u and v, respectively.

1. Introduction

The studies on the edge disjoint paths come up naturally when analyzing connectivity questions or generalizing (integral) network flow problems. Another reason for the grown interest in this area is the variety of applications, e.g. in VLSI-design and interconnection network design. In particular, in the design of a multicomputer (interconnection) system, one important consideration is its fault tolerance, namely its capability of being functional in the presence of failures. The edge connectivity of a connected graph G, denoted by $\lambda(G)$, is the minimum number of edges whose removal from G results in a disconnected graph. The edge connectivity is one of the essential parameters to evaluate the fault tolerance of a network.

To make an overall evaluation on interconnection network with failures, some other measures related to edge connectivity have been studied in recent years. In particular, the extra edge connectivity of hypercubes and folded hypercubes was discussed by several authors in Refs. [4–6,12,13]. In this paper, we consider the classic Menger's Theorem under conditional edge faults.

Theorem 1.1 [7]. *Let x and y be two distinct vertices of a graph G. The minimum size of an x, y edge cut equals the maximum number of edge disjoint x, y-paths.*

Following this theorem, we introduce the F-strong Menger edge connectivity which is similar to the concept on strong Menger connectivity in Oh and Chen [10].

Definition 1.2. A graph G is F-strongly Menger edge connected if for subgraph $G-F$ of G with minimum degree at least 2, $F\subset E(G)$, each pair of vertices u and v in $G-F$ are connected by $min\{deg_{G-F}(u), deg_{G-F}(v)\}$ edge-disjoint fault-free paths in $G-F$, where $deg_{G-F}(u)$ and $deg_{G-F}(v)$ are the degree of u and v in $G-F$, respectively.

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If no confusion should arise, we call a graph strongly Menger edge connected if it is F-strongly Menger edge connected. In [8–10], Oh and Chen proved that an n-dimensional star graph S_n (an n-dimensional hypercube Q_n, respectively) with at most $n-3$ ($n-2$, respectively) vertices removed is strongly Menger connected. Furthermore, Shih et al. [11] provided a result that if we restrict a condition such that each vertex has at least two fault-free adjacent vertices, the hypercube-like graphs still have the strong Menger property, even if there are up to $2n-5$ vertices fault. Here, we show that all hypercubes (folded hypercubes, respectively) is strong Menger edge connectivity if $|F| \leq 2n-4$ ($|F| \leq 2n-2$, respectively) edges fault.

2. Preliminary

Due to attractive topological properties, hypercube has been one of the most fundamental interconnection networks. The hypercube Q_n (with $n \geq 2$) is defined as having the vertex set of binary strings of length n. Two vertices are adjacent if and only if their strings differ in exactly 1 bit. So, Q_n is an n-regular graph with 2^n vertices and $n2^{n-1}$ edges.

Basing on the excellent properties of hypercube, a large number of variants have been proposed. One variant has received a great deal of research is folded hypercube, which is obtained by adding an edge to every pair of nodes with complementary address. So, two vertices of folded hypercube, denoted FQ_n, are adjacent if and only if their strings differ in exactly 1 bit or n bits. Moreover, FQ_n is an $(n+1)$-regular graph with 2^n vertices and $(n+1)2^{n-1}$ edges. The interested reader can refer to [1].

For a graph G, let $\lambda(G)$ denote the edge connectivity of G. For a set of edges $F \subset E(G)$, let $G-F$ denote the graph obtained by deleting F from G. For a set of vertices $S \subset V(G)$, let $N'_G(S)$ be the set of edges with exactly one end in S, and $G[S]$ denote the subgraph induced by S. For brevity, for a vertex u of G, we write $N'_G(\{u\})$ as $N'_G(u)$. For a graph G and a vertex $u \in V(G)$, we denote all adjacent vertices of u in G by $N_G(u)$ and the degree of u in G by $deg_G(u)$. Let u, v be two vertices of G, we use $d_G(u, v)$ to represent the distance of u and v. Other fundamental graph-theoretical terminology, the reader is suggested to refer to [3].

Let S_0 (respectively, S_1) denote the set of all the vertices of Q_n which take on value 0 (respectively, 1) on the ith bit position for some i, $1 \leq i \leq n$. Let $G_0 = G[S_0]$, $G_1 = G[S_1]$, then G_0, G_1 are both isomorphic to Q_{n-1}, and every vertex of G_0 has exactly one neighbor in G_1. Let $M = \{uv \in E(Q_n) | u \in S_0, v \in S_1\}$ be the perfect matching between G_0 and G_1. We use $G_0 \oplus_M G_1$ to denote Q_n. In addition, we sometimes write G_0 and G_1 as Q^{i0}_{n-1} and Q^{i1}_{n-1}, respectively.

By an easy observation, Q_n and FQ_n have the same vertex set. The Hamming distance, denoted by $d_H(u, v)$, between any two vertices u and v of FQ_n is the number of different positions between the binary strings of u and v. It is easy to see that two vertices u and v of folded hypercube FQ_n are adjacent if and only if $d_H(u, v) = 1$ or n. In what follows, we represent $\bar{x_1}\bar{x_2}\dots\bar{x_n}$ and $x_1\dots x_{i-1}\bar{x_i}x_{i+1}\dots x_n$ as $\bar{u}$ and u_i, respectively, where $\bar{x_i} = 1 - x_i$ for $1 \leq i \leq n$. In addition, we write $(u_i)_j$ as u_{ij} for $i \neq j$. Let $PM_i = \{(u, u_i) | u \in V(FQ_n)\}$, $PM = \{(u, \bar{u}) | u \in V(FQ_n)\}$. Clearly, G_0 and G_1 mentioned before are also subgraphs of FQ_n, and in FQ_n, they are connected by two specific perfect matchings PM and PM_i. Then, we denote FQ_n as $G_0 \otimes_{PM \cup PM_i} G_1$. We need the following result in the proof of this paper.

Theorem 2.1 [2]. $\lambda(Q_n) = n$.

In the following, we discuss the strong Menger edge connectivity of hypercubes and folded hypercubes.

3. Strong Menger edge connectivity with conditional faults of hypercubes

In this section, we shall show a main result that an n-dimensional hypercube is F-strong Menger edge connected if $|F| \leq 2n-4$. We need the following lemmas.

Lemma 3.1. *Let $S \subset E(Q_n)$ be a set of edges with $|S| \leq 2n-3$, for $n \geq 2$. There exists a connected component C in $Q_n - S$ with $|V(C)| \geq 2^n - 1$.*

Proof. By induction on n. It is easy to see that the result holds for $n = 2$ and $n = 3$. Assume the lemma holds for $n-1$, $n \geq 4$, we now show that it is true for n.

We may decompose Q_n to $G_0 \oplus_M G_1$. Let S be a set of edges with $|S| \leq 2n-3$, for $n \geq 2$, and let $S_0 = S \cap E(G_0)$, $S_1 = S \cap E(G_1)$, $S_2 = S \cap M$. Then $|S_0| + |S_1| + |S_2| = |S| \leq 2n-3$. Without loss of generality, we assume that $|S_0| \leq |S_1|$. Let C be the largest connected component of $Q_n - S$. It is impossible that both $|S_0|$ and $|S_1|$ are more than $2n-5$. In fact, if $|S_0| > 2n-5$ and $|S_1| > 2n-5$, then $|S| \geq 4n-8$, which contradicts to $|S| \leq 2n-3$. We then consider the following two cases.

Case 1. $|S_0| \leq 2n-5$ and $|S_1| \leq 2n-5$.

It is impossible that both $|S_0|$ and $|S_1|$ are more than $n-2$. In fact, if $|S_0| > n-2$ and $|S_1| > n-2$, then $|S| \geq 2n-2$, which contradicts to $|S| \leq 2n-3$.

Subcase 1a. $|S_0| \leq n-2$ and $|S_1| \leq n-2$.

As $\lambda(G_0) = \lambda(G_1) = n-1$, then $G_0 - S_0$, $G_1 - S_1$ are connected. It follows from $|M| = 2^{n-1}$ and $|S_2| \leq 2n-3$ that $|M| > |S_2|$ for $n \geq 4$. Then $G_0 - S_0$ is connected to $G_1 - S_1$, that is , $Q_n - S$ is connected. So, $|V(C)| = |V(G_0 - S_0)| + |V(G_1 - S_1)| = 2^n$.