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A family of Kurchatov-type methods and its stability

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ABSTRACT

We present a parametric family of iterative methods with memory for solving nonlinear equations, that includes Kurchatov's scheme, preserving its second-order convergence. By using the tools of multidimensional real dynamics, the stability of members of this family is analyzed on low-degree polynomials, showing that some elements of this class have more stable behavior than the original Kurchatov's method. We extend this family to multidimensional case and present different numerical tests for several members of the class on nonlinear systems. The numerical results obtained confirm the dynamical analysis made.

1. Introduction

Computing the solution of nonlinear equations by using iterative methods is considered in this work. In general, the root α of a nonlinear function $f: D \subseteq \mathbb{R} \to \mathbb{R}$ can be obtained as a fixed point of some function ϕ by means of the following fixed-point iteration

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, \dots \tag{1}$$

The most widely used method for this purpose is the classical Newton's method and its derivative-free form known as Steffensen's scheme [25]:

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_k, w_k]}, \quad k = 0, 1, \dots, \tag{2}$$

wherein $w_k = x_k + f(x_k)$ and $f[x_k, w_k] = \frac{f(x_k) - f(w_k)}{x_k - w_k}$. These methods converge quadratically under the conditions that the function f is continuously differentiable and a good initial approximation x_0 is given, [26].

According to the recent trend of researches in this topic, iterative methods with memory (also known as self-accelerating schemes) are worth studying. This is a naming due to Traub [26] when more than one iteration of an scheme, or an updating (using previous iterates) of a free non-zero parameter are applied per cycle so as to calculate the next approximate value.

To review the literature briefly, we remark that optimal Steffensen-type families without memory for solving nonlinear equations were introduced in [22] in a general form, two-step self-accelerating Steffensen-type methods and their

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applications in the solution of nonlinear systems and nonlinear differential equations were discussed in [27]. For further background on this topic, one may consult [4,22].

Due to the fact that (2) suffers of too small local convergence regions [8,10], two-point methods which fall under the definition of schemes with memory are taken into account from time to time. For instance, the secant method is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_{k-1}, x_k]}, \quad k = 1, 2, \dots, \tag{3}$$

with the R-order $\frac{1+\sqrt{5}}{2}$, while the quadratically convergent scheme of Kurchatov is expressed as follows [17]:

$$x_{k+1} = x_k - \frac{f(x_k)}{f[x_{k-1}, 2x_k - x_{k-1}]}, \quad k = 1, 2, \dots \tag{4}$$

Motivated by the recent developments in this area (see, for example, [16]), we here propose a family of parametric methods, variants of (4), preserving the order of convergence of the original scheme. The stability of the different elements of this family is analyzed on low-degree polynomials, showing that some members of the class have good stability properties, meanwhile others present chaotic behavior. Hence, this would be our main motivation in this work.

Kurchatov's scheme is a good derivative-free iterative scheme with memory with the same order of convergence and number of new functional evaluations per iteration as Newton's procedure. We propose a new family of parametric methods with memory, including Kurchatov's one, with the same order for any value of the parameter. By using real multidimensional dynamical tools, we analyze the stability of the elements of this class, finding many members as stable as the original one and also other with several pathologies and chaotical behavior, strange attractors, period-doubling bifurcations, etc., that must be avoided in practical applications. This family is extended to the multidimensional case and the advantages and drawbacks found in the dynamical study are observed in the numerical tests made.

The paper is divided into several sections and is organized as follows. After this introductory review, a family of Kurchatov-type schemes is proposed and studied in Section 2. The drawbacks and strong points are also pointed out. Section 3 includes the analysis of the stability by providing the dynamical behavior of these schemes. In Section 4, a generalization of the proposed variant for systems of nonlinear equations is brought forward and some numerical examples are considered to confirm the theoretical results. Concluding remarks are given in Section 5.

2. Methodology and convergence analysis

One technique to extend the already known schemes and provide some generalizations of them is due to imposing a parameter into their structure so as to keep the convergence R-order and improve it to some extend under some considerations.

Let us re-write the extra point involved in (4) as $2x_k - x_{k-1} = x_k + x_k - x_{k-1}$. The increment $x_k - x_{k-1}$ would roughly be equal to $f(x_k)$ in the convergence phase. On the other hand, in (2), we have $w_k = x_k + \beta f(x_k)$, so we could apply a similar idea here for (4) to impose a free nonzero parameter β into the structure in what follows:

$$\begin{cases} l_k = \beta x_{k-1}, \quad w_k = x_k + \beta(x_k - x_{k-1}), \\ f[w_k, l_k] = \dfrac{f(w_k) - f(l_k)}{w_k - l_k}, \\ x_{k+1} = x_k - \dfrac{f(x_k)}{f[w_k, l_k]}, \quad k = 1, 2, \dots \end{cases} \tag{5}$$

Theorem 1. *Let function $f(x)$ be differentiable enough in a neighborhood of its simple zero α. If initial approximations x_0 and x_1 are sufficiently close to α then, convergence R-order of the improved Kurchatov's method with memory (5) is two.*

Proof. The R-quadratic convergence of (5) can be proved with the help of the majorants of Kantorovich (see e.g. [5]). In this paper, we introduce a different procedure to prove the R-quadratic convergence, where we show a shorter alternative analytic proof. To this end, let the sequence $\{x_k\}$ be defined by (5) and denote as $e_k = x_k - \alpha$ and $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$, $j \geq 2$. Using Taylor expansion, we have

$$f(x_k) = f'(\alpha)[e_k + c_2 e_k^2 + c_3 e_k^3 + c_4 e_k^4 + c_5 e_k^5] + O(e_k^5), \tag{6}$$

and

$$\begin{aligned} f[w_k, l_k] = {} & (f'(\alpha) + c_2 f'(\alpha)(1+\beta)e_k + c_3 f'(\alpha)(1+\beta))^2 e_k^2 + O(e_k^3)) \\ & + (-c_3 f'(\alpha)\beta(1+\beta)e_k - 2(c_4 f'(\alpha)\beta(1+\beta)^2)e_k^2 + O(e_k^3))e_{k-1} \\ & + (c_3 f'(\alpha)\beta^2 + 2c_4 f'(\alpha)\beta^2(1+\beta)e_k + 4c_5 f'(\alpha)\beta^2(1+\beta)^2 e_k^2 + O(e_k^3))e_{k-1}^2 + O(e_{k-1}^3). \end{aligned} \tag{7}$$

Combining (6) and (7) into (5), one gets that

$$\begin{aligned} e_{k+1} = {} & (c_2\beta e_k^2 + O(e_k^3)) + (-c_3\beta(1+\beta)e_k^2 + O(e_k^3))e_{k-1} \\ & + (c_3\beta^2 e_k + \beta^2(2c_4(1+\beta) - c_2(c_3 + 2c_3\beta))e_k^2 + O(e_k^3))e_{k-1}^2 + O(e_{k-1}^3). \end{aligned} \tag{8}$$