Applied Mathematics and Computation 294 (2017) 343–352

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

New methods for computing the Drazin-inverse solution of singular linear systems

F. Toutounian [a,b,*], R. Buzhabadi [a]

[a] Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran
[b] The Center of Excellence on Modeling and Control Systems, Ferdowsi University of Mashhad, Mashhad, Iran

ARTICLE INFO

Keywords:
Drazin-inverse solution
DGMRES method
LGMRES method
GMRES-E method
Singular systems
Error vector

ABSTRACT

The DGMRES method is an iterative method for computing the Drazin-inverse solution of consistent or inconsistent linear systems of the form $Ax = b$, where $A \in \mathbb{C}^{n\times n}$ is a singular and in general non-Hermitian matrix that has an arbitrary index. This method is generally used with restarting. But the restarting often slows down the convergence and DGMRES often stagnates. Based on the LGMRES and GMRES-E methods, we present two new techniques for accelerating the convergence of restarted DGMRES by adding some approximate error vectors or approximate eigenvectors (corresponding to a few of the smallest eigenvalues) to the Krylov subspace. We derive the implementation of these methods and present some numerical examples to show the advantages of these methods.

1. Introduction

In this paper, we consider the problem of finding a solution of the system

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a singular matrix, $b \in \mathbb{R}^n$ and ind(A) is arbitrary. Here ind(A), the index of A is the size of the largest Jordan block corresponding to zero eigenvalue of A. We recall that the Drazin-inverse solution of the system (1) is the vector $A^D b$, where A^D is the Drazin-inverse of the singular matrix A [3,5]. The Drazin-inverse solution $A^D b$ is the unique solution of the equation $A^{a+1}x = A^a b$ that belongs to $\mathcal{R}(A^a)$ [24].

The Drazin-inverse has various applications in the theory of finite Markov chains [5], the study of singular differential and difference equations [5], the investigation of Cesaro–Neumann iterations [12], cryptographic system [11], iterative methods in numerical analysis [8,9], multibody system dynamics [22], and others.

The problem of finding the solution of the form $A^D b$ for (1) is very common in the literature and many different techniques have been developed in order to solve it [7–10,19–21,23,25].

In [19], Sidi developed the DGMRES method for singular systems that is analogous to the GMRES method for non-singular systems. In addition, in [19], the author proposed an effective mode of usage for DGMRES, denoted DGMRES(m), which is analogous to the GMRES(m) and requires a fixed amount of storage for its implementation. In restarted DGMRES (DGMRES(m)) the method is restarted once the Krylov subspace reaches dimension m, and the current approximate solution

* Corresponding author.
E-mail addresses: toutouni@math.um.ac.ir (F. Toutounian), reza_bozhabadi@yahoo.com, reza.bozhabadi@gmail.com (R. Buzhabadi).

http://dx.doi.org/10.1016/j.amc.2016.09.013
0096-3003/

becomes the new initial guess for the next m iterations. The restart parameter m is generally chosen small relative to n to keep storage and computation requirements reasonable.

In general, restarting slows the convergence of GMRES and DGMRES methods. However, it is possible to save some important information at the time of the restart and used in the next cycle. For accelerating the convergence of the restarted GMRES method, Chapman and Saad [6] and Morgan [14–16] proposed augmentation of Krylov subspaces generated by restarted GMRES by spaces spanned by certain eigenvectors or Ritz vectors. Baglama and Reichel [2] proposed to augment Krylov subspaces generated by GMRES by linear spaces that are not defined by Ritz vectors. They show that when the linear system of equations arises from the discretization of a well-posed problem, augmentation can reduce the number of iterations required to determine an approximate solution of desired accuracy. Baker et al. [1] served the error approximation for augmenting the next approximation space and presented the LGMRES algorithm. They discussed some of its properties and illustrated that the LGMRES can significantly accelerate the convergence of restarted GMRES method.

In this paper, we propose two new methods LDGMRES(m, k) and DGMRES-E(m, k) which are analogous to the LGMRES(m, k) and the GMRES-E(m, k), respectively [1,14]. For accelerating the convergence of the DGMRES(m), LDGMRES(m, k) augments the next Krylov subspace with some approximate error vectors and DGMRES-E(m, k) includes the approximate eigenvectors determined from the previous subspace in the new subspace. By numerical examples, we show the advantages of these methods.

The paper is organized as follows. In Section 2, we will give a review of the DGMRES algorithm. In Section 3, we will derive the LDGMRES(m, k) algorithm by describing an augmented DGMRES algorithm. In Section 4, we will develop the DGMRES-E(m, k) algorithm. In Section 5, the results of some numerical examples are given. Section 6 is devoted to concluding remarks.

2. DGMRES algorithm

DGMRES method is a Krylov subspace method for computing the Drazin-inverse solution of consistent or inconsistent linear systems (1) [19,20]. In this method, there are not any restriction on the matrix A. Thus, in general, A is non-Hermitian, $a = \mathrm{ind}(A)$ is arbitrary, and the spectrum of A can have any shape. Thus, it is unnecessary for us to put any restriction on the linear system $Ax = b$. The system may be consistent or inconsistent. We only assume that ind(A) is known.

DGMRES starts with an initial vector x_0 and generates a sequence of vectors $x_0, x_1, \ldots,$ as

$$x_m = x_0 + \sum_{i=1}^{m-a} c_i A^{a+i-1} r_0, \quad r_0 = b - Ax_0.$$

Then

$$r_m = b - Ax_m = r_0 - \sum_{i=1}^{m-a} c_i A^{a+i} r_0.$$

The Krylov subspace we will use is

$$\mathcal{K}_{m-a}(A, A^a r_0) = \mathrm{span}\{A^a r_0, A^{a+1} r_0, \ldots, A^{m-1} r_0\}.$$

The vector x_m produced by DGMRES satisfies

$$\|A^a r_m\| = \min_{x \in x_0 + \mathcal{K}_{m-a}(A, A^a r_0)} \|A^a (b - Ax)\|_2. \tag{2}$$

The implementation of the DGMRES method is given by Algorithm 1. More details about the DGMRES algorithm can be found in [20].

In our discussion, we refer to the group of m iterations between successive restarts as a cycle. The restart number is denoted with a subscript: r_i is the residual after i cycle or $m \times i$ iterations. During each restart cycle i DGMRES(m) finds $x_{i+1} \in x_i + \mathcal{K}_{m-a}(A, A^a r_i)$ such that $A^a r_{i+1} \perp A^{a+1} \mathcal{K}_{m-a}(A, A^a r_i)$.

In the sequel, some MATLAB notation is used; for instance, $\widetilde{H}_{m-a}(m-a+1 : m+1, 1 : m-a)$ denotes the portion of $\widetilde{H}_{m-a}$ with rows from $m-a+1$ to $m+1$ and columns from 1 to $m-a$. In addition, we denote a matrix X with l columns by X_l.

3. A new algorithm: LDGMRES

When an iterative approach is restarted, the current approximation space is discarded at each restart. Our technique attempts to accelerate the convergence of DGMRES(m) by retaining some of the information that is typically discarded at the time of restart.

Suppose that x is the true solution to the problem (1). The error after the ith restart cycle of DGMRES(m) is denoted by e_i, where

$$e_i \equiv x - x_i. \tag{3}$$

If our approximation space contains the exact correction e_i such that $x = x_i + e_i$, then we have solved the problem. We define

$$z_i \equiv x_i - x_{i-1}, \tag{4}$$