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Oscillation-free solutions to Volterra integral and integro-differential equations with periodic force terms

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ABSTRACT

Volterra equations governed by oscillatory functions arise frequently in applied fields. To construct efficient algorithms for solving these equations, oscillatory properties of their solutions should be studied. In this paper, oscillatory orders of solutions to a class of highly oscillatory Volterra integral equations are presented. Then some modified solutions are given with the help of asymptotic expansions of highly oscillatory integrals. Finally, theoretical analysis verifies these modified solutions enjoy less oscillation than the original one, and extensions to highly oscillatory Volterra integro-differential equations are considered.

1. Introduction

Volterra integral and integro-differential equations is a classic subject in applied mathematics, and enjoys wide applications in practical problems. For example, the problem of finding the optimal harvesting effort leads to a Volterra integral equation [1]. In addition, many special functions, such as generalized hypergeometric functions, can be calculated by solving a class of fractional differintegral equations [2]. For more comprehensive analyses of VIEs and VIDEs, one can refer to [3–5].

Highly oscillatory Volterra integral and integro-differential equations are two kinds of important Volterra equations. More and more attention, both theoretically and numerically, is paid to them recently. In this paper, we mainly consider the following highly oscillatory Volterra integral equations (HOVIEs),

$$u(t) = f(t)e^{i\omega g(t)} + (\mathscr{V}u)(t), \ \ t \in I := [0, T], \tag{1}$$

with Volterra integral operator defined by

$$(\mathscr{V}\varphi)(t) := \int_0^t K(t, s)\varphi(s)ds. \tag{2}$$

Here the frequency $\omega \gg 1$, which introduces a highly oscillatory term in the force function $f(t)e^{i\omega g(t)}$.

In the literature, oscillatory properties of solutions of integral equations attracted many researchers' attention. Ursell gave the oscillatory property of a class of Fredholm integral equations by splitting the Fredholm integral operator into two highly oscillatory Volterra integral operators [6]. In [7], a hybrid numerical-asymptotic method for solving boundary integral equations with highly oscillatory kernels is developed by applying the asymptotic property of solutions. Brunner's work [8] reawakened researchers' interest in this field. In his paper, compact and noncompact cordial Volterra integral operators with highly oscillatory kernels were studied and their high-oscillation properties were built by investigating highly

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oscillatory integrals (HOIs). Following Brunner's methodology, Wang and Xu [9], and Brunner, Ma and Xu [10] analyzed the oscillation of solutions of integral equations with highly oscillatory kernels based on structured oscillatory spaces. In [11], Xiang gave the asymptotic expansions of solutions of the first kind VIE with highly oscillatory Bessel kernels by Laplace transform.

On the other hand, studies on numerically solving HOVIEs are also prevalent in recently years. Brunner, Davis and Duncan investigated the discontinuous Galerkin method for the first kind Volterra equations [12]. Numerical experiments showed this approximation was effective in solving a VIE arising in the scattering problem. For the same equation, Wang and Xiang presented an efficient quadrature method by employing the Clenshaw-Curtis Filon-type algorithm for calculation of HOIs [13]. Moreover, Xiang and Brunner studied Filon collocation methods for VIEs with highly oscillatory Bessel kernels [14]. And convergence rates were presented by analyzing asymptotic properties of HOIs.

According to theoretical analyses of Filon collocation methods for HOVIEs(see [15–17]), derivatives of solutions to (1) play an important role in controlling frequency-based convergence rates of numerical methods. This inspires the study on highly oscillatory properties of solutions in this paper. In the following section, some oscillation-free solutions to HOVIEs are introduced. Then similar techniques are applied to HOVIDEs in Section 3.

2. Oscillation of solutions to Volterra integral equations

We firstly recall the definition of oscillatory order.

Definition 1 (also see [9,10]). An oscillatory function $\varphi(\cdot)$ in the continuous function space $C([a, b])$ with norm $\|\cdot\|_\infty$ is said to be $\omega-$oscillatory of order α if α is the integer for which there exists a positive constant c independent of ω [1], such that for all $\omega > 1$,

$$\omega^{-\alpha}\|\varphi\|_\infty \le c. \tag{3}$$

Then let us introduce a famous lemma about the generalized Fourier transform, which can be found in [18, pp.333].

Lemma 1. *Suppose $g(\cdot)$ is real-valued and smooth in (a, b), and that $|g^{(k)}(s)| \ge 1$ for all $s \in (a, b)$ for a fixed value of k. Moreover, suppose $\phi(\cdot) \in C^1(a, b)$ and $\phi'(\cdot) \in L[a, b]$. We can conclude that*

$$\left|\int_a^b e^{i\omega g(s)}\phi(s)ds\right| = O(\omega^{-1/k}),\ \ \omega \to \infty. \tag{4}$$

Now we can obtain the oscillatory order of solutions to (1).

Theorem 1. *For any positive integer n, suppose $f(\cdot) \in C^n(I)$ and $g(\cdot) \in C^n(I)$ in (1), and $g(\cdot)$ is strictly increasing. Then the $n-$th derivative of $u(\cdot)$ is $\omega-$oscillatory of order n.*

Proof. According to [5], the solution $u(\cdot)$ to (1) can be rewritten in the resolvent form,

$$u = \sum_{j=0}^{\infty}\mathscr{V}^j(f\cdot p_g), \tag{5}$$

where $p_g(\cdot) = e^{i\omega g(\cdot)}$. It can be easily seen that

$$\|\mathscr{D}^n(f\cdot p_g)\|_\infty = O(\omega^n),\ \ \omega \to \infty, \tag{6}$$

where $\mathscr{D}$ denotes the differential operator. With the help of Lemma 1, we find

$$\|\mathscr{D}^n\mathscr{V}^m(f\cdot p_g)\|_\infty \le \begin{cases} c, & 0 < n \le m, \\ c\omega^{n-m}, & 0 < m < n. \end{cases} \tag{7}$$

This completes the proof. □

Theorem 1 implies oscillatory orders of derivatives of the solution will increase greatly as the frequency goes to infinity, which may introduce intolerable errors in numerical solutions. Therefore, high precision algorithms for HOVIEs should be constructed based on eliminating the oscillation order of solutions. Here we introduce some modified solutions to (1) and give their oscillatory properties.

Definition 2. $\mathscr{I}_n$ is a set of $n-$dimensional vectors, whose elements are integer, and for any $\alpha = (\alpha_j)_{1\le j\le n} \in \mathscr{I}_n$, it follows that

$$0 \le \alpha_k \le n,\ \ k = 1, 2, \ldots, n,$$
$$\alpha_{i+1} = 0,\ \ \text{if } \alpha_i = 0,\ \ i = 1, 2, \ldots, n-1.$$

[1] In this paper, we always assume 'c' be a constant independent of ω, although its value varies in different places.