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On the asymptotic expansions of products related to the Wallis, Weierstrass, and Wilf formulas

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ABSTRACT

For all integers $n \geq 1$, let

$$W_n(p,q) = \prod_{j=1}^{n} \left\{ e^{-p/j} \left(1 + \frac{p}{j} + \frac{q}{j^2} \right) \right\}$$

and

$$R_n(p,q) = \prod_{j=1}^{n} \left\{ e^{-p/(2j-1)} \left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2} \right) \right\},$$

where p, q are complex parameters. The infinite product $W_\infty(p, q)$ includes the Wallis and Wilf formulas, and also the infinite product definition of Weierstrass for the gamma function, as special cases. In this paper, we present asymptotic expansions of $W_n(p, q)$ and $R_n(p, q)$ as $n \to \infty$. In addition, we also establish asymptotic expansions for the Wallis sequence.

1. Introduction

The famous Wallis sequence W_n, defined by

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2 - 1} \qquad (n \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

has the limiting value

$$W_\infty = \prod_{k=1}^{\infty} \frac{4k^2}{4k^2 - 1} = \frac{\pi}{2} \tag{1.2}$$

established by Wallis in 1655; see [5, p. 68]. Several elementary proofs of this well-known result can be found in [3,23,37]. An interesting geometric construction that produces the above limiting value can be found in Myerson [30]. Many formulas exist for the representation of π, and a collection of these formulas is listed [33,34]. For more history of π see [2,4,5,14].

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The following infinite product definition for the gamma function is due to Weierstrass (see, for example, [1, p. 255, Entry (6.1.3)]):

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left\{ e^{-z/n}\left(1 + \frac{z}{n}\right)\right\}, \tag{1.3}$$

where γ denotes the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.5772156649\ldots.$$

In 1997, Wilf [39] posed the following elegant infinite product formula as a problem:

$$\prod_{j=1}^{\infty} \left\{ e^{-1/j}\left(1 + \frac{1}{j} + \frac{1}{2j^2}\right)\right\} = \frac{e^{\pi/2} + e^{-\pi/2}}{\pi e^{\gamma}}, \tag{1.4}$$

which contains three of the most important mathematical constants, namely π, e and γ. Subsequently, Choi and Seo [12] proved (1.4), together with three other similar product formulas, by making use of well-known infinite product formulas for the circular and hyperbolic functions and the familiar Stirling formula for the factorial function.

In 2003, Choi et al. [11] presented the following two general infinite product formulas, which include Wilf's formula (1.4) and other similar formulas in Choi and Seo [12] as special cases:

$$\prod_{j=1}^{\infty} \left\{ e^{-1/j}\left(1 + \frac{1}{j} + \frac{\alpha^2 + 1/4}{j^2}\right)\right\} = \frac{2(e^{\pi\alpha} + e^{-\pi\alpha})}{(4\alpha^2 + 1)\pi e^{\gamma}} \qquad \left(\alpha \in \mathbb{C};\ \alpha \neq \pm\frac{1}{2}i\right) \tag{1.5}$$

and

$$\prod_{j=1}^{\infty} \left\{ e^{-2/j}\left(1 + \frac{2}{j} + \frac{\beta^2 + 1}{j^2}\right)\right\} = \frac{e^{\pi\beta} - e^{-\pi\beta}}{2\beta(\beta^2 + 1)\pi e^{2\gamma}} \qquad (\beta \in \mathbb{C} \setminus \{0\};\ \beta \neq \pm i), \tag{1.6}$$

where $i = \sqrt{-1}$ and $\mathbb{C}$ denotes the set of complex numbers. In 2013, Chen and Choi [7] presented a more general infinite product formula that included the formulas (1.5) and (1.6) as special cases:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/j}\left(1 + \frac{p}{j} + \frac{q}{j^2}\right)\right\} = \frac{e^{-p\gamma}}{\Gamma\left(1 + \frac{1}{2}p + \frac{1}{2}\Delta\right)\Gamma\left(1 + \frac{1}{2}p - \frac{1}{2}\Delta\right)} \tag{1.7}$$

and also another interesting infinite product formula:

$$\prod_{j=1}^{\infty} \left\{ e^{-p/(2j-1)}\left(1 + \frac{p}{2j-1} + \frac{q}{(2j-1)^2}\right)\right\} = \frac{2^{-p}\pi e^{-p\gamma/2}}{\Gamma\left(\frac{1}{2} + \frac{1}{4}p + \frac{1}{4}\Delta\right)\Gamma\left(\frac{1}{2} + \frac{1}{4}p - \frac{1}{4}\Delta\right)}, \tag{1.8}$$

where $p, q \in \mathbb{C}$ and $\Delta := \sqrt{p^2 - 4q}$.

The formula (1.7) can be seen to include the formulas (1.2)–(1.6) as special cases. By setting $(p, q) = (0, -1/4)$ in (1.7), we have

$$\prod_{j=1}^{\infty} \left(1 - \frac{1}{4j^2}\right) = \frac{2}{\pi}, \tag{1.9}$$

whose reciprocal becomes the Wallis product (1.2). Also setting $q = 0$ in (1.7), we obtain

$$\prod_{j=1}^{\infty} \left\{ e^{-p/j}\left(1 + \frac{p}{j}\right)\right\} = \frac{e^{-p\gamma}}{\Gamma(p+1)}. \tag{1.10}$$

Noting that $\Gamma(z+1) = z\Gamma(z)$ and replacing p by z in (1.10) we recover the Weierstrass formula (1.3). Setting $(p, q) = (1, 1/2)$ in (1.7) yields the Wilf formula (1.4) and setting

$$(p, q) = \left(1, \alpha^2 + \frac{1}{4}\right) \quad \text{and} \quad (p, q) = \left(2, \beta^2 + 1\right) \tag{1.11}$$

in (1.7) yields the formulas (1.5) and (1.6), respectively.

With $(p, q) = (-1, 1/4)$ in (1.7), we obtain the beautiful infinite product formula expressed in terms of the most important constants π, e and γ, namely

$$\prod_{j=1}^{\infty} \left\{ e^{1/j}\left(1 - \frac{1}{2j}\right)^2\right\} = \frac{e^{\gamma}}{\pi}. \tag{1.12}$$