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Degree reduction of composite Bézier curves

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ABSTRACT

This paper deals with the problem of multi-degree reduction of a composite Bézier curve with the parametric continuity constraints at the endpoints of the segments. We present a novel method which is based on the idea of using constrained dual Bernstein polynomials to compute the control points of the reduced composite curve. In contrast to other methods, ours minimizes the L_2-error for the whole composite curve instead of minimizing the L_2-errors for each segment separately. As a result, an additional optimization is possible. Examples show that the new method gives much better results than multiple application of the degree reduction of a single Bézier curve.

1. Introduction

In recent years, many methods have been used to reduce the degree of Bézier curves with constraints (see, e.g., [1–6,8,9,11,13–20]). Most of these papers give methods of multi-degree reduction of a *single* Bézier curve with constrains of endpoints (parametric or geometric) continuity of arbitrary order with respect to L_2-norm. Observe, however, that degree reduction schemes often need to be combined with the subdivision algorithm, i.e., a high degree curve is replaced by a number of lower degree curve segments, or a *composite* Bézier curve, and continuity between adjacent lower degree curve segments should be maintained. Intuitively, a possible approach in such a case is applying the multi-degree reduction procedure to one segment of the curve after another with properly chosen endpoints continuity constraints. However, in general, the obtained solution does not minimize the distance between two composite curves.

In this paper, we give the optimal least-squares solution of multi-degree reduction of a composite Bézier curve with the parametric continuity constraints at the endpoints of the segments. More specifically, we consider the following approximation problem.

Problem 1.1 (C^r-constrained multi-degree reduction of a composite Bézier curve)**.** Let $a = t_0 < t_1 < \cdots < t_s = b$ be a partition of the interval $[a, b]$. Let there be given a degree $\boldsymbol{n} = (n_1, n_2, \ldots, n_s)$ composite Bézier curve $P(t)$ $(t \in [a, b])$ in $\mathbb{R}^d$ that in the interval $[t_{i-1}, t_i]$ $(i = 1, 2, \ldots, s)$ is exactly represented as a Bézier curve $P_i(u)$ $(u \in [0, 1])$ of degree n_i, i.e.,

$$P(t) = P_i(u) := \sum_{j=0}^{n_i} p_{i,j} B_j^{n_i}(u) \qquad (t_{i-1} \le t \le t_i;\ u := (t - t_{i-1})/h_i),$$

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where $h_i := t_i - t_{i-1}$, and

$$B_j^n(u) := \binom{n}{j} u^j (1-u)^{n-j} \qquad (0 \le j \le n)$$

are the Bernstein basis polynomials of degree n. Find a composite Bézier curve $Q(t)$ ($t \in [a, b]$) of degree $\boldsymbol{m} = (m_1, m_2, \ldots, m_s)$ which in the interval $[t_{i-1}, t_i]$ ($i = 1, 2, \ldots, s$) is exactly represented as a Bézier curve $Q_i(u)$ ($u \in [0, 1]$) of degree $m_i < n_i$, i.e.,

$$Q(t) = Q_i(u) := \sum_{j=0}^{m_i} q_{i,j} B_j^{m_i}(u) \qquad (t_{i-1} \le t \le t_i;\ u := (t - t_{i-1})/h_i), \tag{1.1}$$

such that the squared L_2-error

$$E := \int_a^b \|P(t) - Q(t)\|^2 \, \mathrm{d}t = \sum_{i=1}^{s} E_i, \tag{1.2}$$

where

$$E_i := \int_{t_{i-1}}^{t_i} \|P(t) - Q(t)\|^2 \, \mathrm{d}t = h_i \int_0^1 \|P_i(u) - Q_i(u)\|^2 \, \mathrm{d}u,$$

reaches the minimum under the additional conditions that

$$\left.\frac{\mathrm{d}^j Q(t)}{\mathrm{d}t^j}\right|_{t=t_0} = \left.\frac{\mathrm{d}^j P(t)}{\mathrm{d}t^j}\right|_{t=t_0} \qquad (j = 0, 1, \ldots, r_0), \tag{1.3}$$

$$\left.\frac{\mathrm{d}^j Q(t)}{\mathrm{d}t^j}\right|_{t=t_i-} = \left.\frac{\mathrm{d}^j Q(t)}{\mathrm{d}t^j}\right|_{t=t_i+} \qquad (i = 1, 2, \ldots, s-1;\ j = 0, 1, \ldots, r_i), \tag{1.4}$$

$$\left.\frac{\mathrm{d}^j Q(t)}{\mathrm{d}t^j}\right|_{t=t_s} = \left.\frac{\mathrm{d}^j P(t)}{\mathrm{d}t^j}\right|_{t=t_s} \qquad (j = 0, 1, \ldots, r_s), \tag{1.5}$$

where $r_j \ge 0$ ($j = 0, 1, \ldots, s$) and $r_{i-1} + r_i < m_i - 1$ ($i = 1, 2, \ldots, s$). We will say that the curves P and Q satisfy the $C^{\boldsymbol{r}}$*–continuity conditions* at the points $t_0, t_1, \ldots, t_s$, where we use the notation $\boldsymbol{r} := (r_0, r_1, \ldots, r_s)$. Here $\|\cdot\|$ is the Euclidean vector norm.

Remark 1.2. One may think that the conditions

$$\left.\frac{\mathrm{d}^j Q(t)}{\mathrm{d}t^j}\right|_{t=t_i} = \left.\frac{\mathrm{d}^j P(t)}{\mathrm{d}t^j}\right|_{t=t_i} \qquad (i = 1, 2, \ldots, s-1;\ j = 0, 1, \ldots, r_i)$$

would be more natural than (1.4). However, in contrast to our new method, such an approach leaves no room for additional optimization.

Remark 1.3. Sometimes, it may be useful to interpolate the endpoints of the original segments (see Example 4.2), i.e., to demand that $Q(t_i) = P(t_i)$ holds for $i = 1, 2, \ldots, s-1$. In such a case, constraints (1.4) should be appropriately modified by restricting the range of j to $1, 2, \ldots, r_i$.

The paper is organized as follows. In Section 2, we recall some results which are later applied in the solution of the problem, given in Section 3. Several illustrative examples are presented in Section 4. Finally, Section 5 contains some concluding remarks.

We end this section with introducing some notation. The *shifted factorial* is defined by $(c)_0 := 1$, $(c)_j := c(c+1)\cdots(c+j-1)$ ($j = 1, 2, \ldots$). The *iterated forward difference operator* Δ^j is given by

$$\Delta^j \alpha_k := \Delta^{j-1}\alpha_{k+1} - \Delta^{j-1}\alpha_k \quad (j = 1, 2, \ldots) \quad \text{and} \quad \Delta^0 \alpha_k := \alpha_k.$$

Moreover, we adopt the convention that in an expression of the form $\Delta^j \gamma_{i,k}$ the operator Δ^j acts on the second variable (first variable being fixed), e.g.,

$$\Delta^2 \gamma_{i,k} = \Delta\gamma_{i,k+1} - \Delta\gamma_{i,k} = \gamma_{i,k+2} - 2\gamma_{i,k+1} + \gamma_{i,k}.$$

2. Preliminaries

Let us denote by $\Pi_m^{(k,l)} = \operatorname{span}\{B_{k+1}^m, B_{k+2}^m, \ldots, B_{m-l-1}^m\}$, where k and l are natural numbers such that $k + l < m - 1$, the space of all polynomials of degree at most m, whose derivatives of order $\le k$ at $t = 0$ and of order $\le l$ at $t = 1$ vanish. There is a unique *dual constrained Bernstein basis of degree* m (see, e.g., [10]), $D_{k+1}^{(m,k,l)}, D_{k+2}^{(m,k,l)}, \ldots, D_{m-l-1}^{(m,k,l)}$, satisfying $\langle D_j^{(m,k,l)}, B_h^m \rangle =$