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Approximation by (p, q)-Baskakov–Beta operators

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ABSTRACT

In the present paper, we consider (p, q)-analogue of the Baskakov–Beta operators and using it, we estimate some direct results on approximation. Also, we represent the convergence of these operators graphically using MATLAB.

1. Introduction

Approximation theory has been an engaging field of research with abstract approximation to the core (cf. [15]). Varied operators with their approximation properties, mainly the quantitative one, have been discussed and studied by many researchers. It has been seen that the generalizations of several well-known operators to quantum-calculus (q-calculus) were introduced in the last three decades and their approximation behavior were also discussed (see [3,10–12]). Further generalization of quantum variant is the post-quantum calculus, denoted by (p, q)-calculus. Very recently, some researchers studied in this direction (see [4,9,17]). Few basic definitions and notations mentioned below may be found in these papers and references therein.

The (p, q)-numbers are given by

$$[n]_{p,q} := p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \begin{cases} \dfrac{p^n - q^n}{p-q}, & \text{if} p \neq q \neq 1; \\ n, & \text{if } p = q = 1. \end{cases}$$

The (p, q)-factorial is given by $[n]_{p,q}! = \prod_{r=1}^{n} [r]_{p,q}, \quad n \geqslant 1, \quad [0]_{p,q}! = 1$. The (p, q)-binomial coefficient satisfies

$$\begin{bmatrix} n \\ r \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-r]_{p,q}![r]_{p,q}!}, \quad 0 \leqslant r \leqslant n.$$

Let n be a non-negative integer, the (p, q)-Gamma function is defined as

$$\Gamma_{p,q}(n+1) = \frac{(p \ominus q)_{p,q}^n}{(p-q)^n} = [n]_{p,q}!, \quad 0 < q < p,$$

where $(p \ominus q)_{p,q}^n = (p-q)(p^2-q^2)(p^3-q^3)\cdots(p^n-q^n)$.

The (p, q)-integral for $0 < q < p \leq 1$ (generalized Jackson integral) is defined as

$$\int_0^a f(x)\, d_{p,q}x = (p-q)a \sum_{i=0}^{\infty} \frac{q^i}{p^{i+1}}\, f\left(\frac{a\, q^i}{p^{i+1}}\right), \quad x \in [0, a]. \tag{1}$$

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By simple computation, we get

$$\int_0^a x^n \, d_{p,q}x = \frac{a^{n+1}}{[n+1]_{p,q}}.$$

Also, the integral (1) includes the nodes $x_i = x_i(p,q) = \frac{a q^i}{p^{i+1}}, \quad i = 0, 1, \ldots,$ geometrically distributed in $(0, +\infty)$, not only in (0, a), as in the case $p = 1$ (standard Jackson's q-integral). Moreover, one may observe that only a finite number of nodes in (1) are outside (0, a), i.e., those x_i for which $q^i > p^{i+1}$. Thus, the above definition of (p, q)-integral may be well utilized to define the (p, q)-extensions of well-known results.

For $m, n \in \mathbb{N}$, the (p, q)-Beta function of second kind considered in [2] is given by

$$B_{p,q}(m,n) = \int_0^\infty \frac{t^{m-1}}{(1 \oplus pt)_{p,q}^{m+n}} \, d_{p,q}t,$$

where the (p, q)-power basis is given by

$$(1 \oplus pt)_{p,q}^{m+n} = (1+pt)(p+pqt)(p^2+pq^2t)\ldots(p^{m+n-1}+pq^{m+n-1}t).$$

Using the (p, q)-integration by parts:

$$\int_a^b f(px) \, D_{p,q}g(x) \, d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx) \, D_{p,q}f(x) \, d_{p,q}x,$$

it was shown in [2] that the following relation is satisfied by the (p, q)-analogues of Beta and Gamma functions:

$$B_{p,q}(m,n) = \frac{q \quad \Gamma_{p,q}(m)\,\Gamma_{p,q}(n)}{\left(p^{m+1} \; q^{m-1}\right)^{m/2} \quad \Gamma_{p,q}(m+n)}.$$

As a special case, if $p = q = 1$, $B(m,n) = \Gamma(m)\,\Gamma(n)/\Gamma(m+n)$. It may be observed that in (p, q)-setting, order is important, which is the reason why (p, q)-variant of Beta function does not satisfy commutativity property, i.e., $B_{p,\,q}(m,\,n) \neq B_{p,\,q}(n,\,m)$.

For $n \in \mathbb{N}$, $x \in [0, \infty)$ and $0 < q < p \leq 1$, the (p, q)-analogue of Baskakov operators can be defined as

$$B_{n,p,q}(f,x) = \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right),$$

where (p, q)-Baskakov basis function is given by

$$b_{n,k}^{p,q}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} \frac{x^k}{(1 \oplus x)_{p,q}^{n+k}}.$$

Gupta [9] considered this form of (p, q)-Baskakov operators while studying its Kantorovich variant. This form was also considered by T. Acar et al. [5].

Remark 1. It has been observed in [9] that the (p, q)-Baskakov operators satisfy the following recurrence relation:

$$[n]_{p,q} T_{n,m+1}^{p,q}(qx) = q\, p^{n-1} \; x\,(1+px) \; D_{p,q}[T_{n,m}^{p,q}(x)] + [n]_{p,q}\, q\, x\, T_{n,m}^{p,q}(qx),$$

where $T_{n,m}^{p,q}(x) := B_{n,p,q}(e_m, x) = \sum_{k=0}^{\infty} b_{n,k}^{p,q}(x) \left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right)^m$.

Then, we have

$$B_{n,p,q}(e_0, x) = 1, \quad B_{n,p,q}(e_1, x) = x,$$

$$B_{n,p,q}(e_2, x) = \frac{[n+1]_{p,q}\, x^2 + p^{n-1}\, q\, x}{q\,[n]_{p,q}},$$

where $e_i(t) = t^i$, $i = 0, 1, 2$. In case $p = 1$, we get the q-Baskakov operators [1,11]. If $p = q = 1$, then these operators reduce to the well-known Baskakov operators.

2. Construction of operators and moments

In the year 1985, Sahai–Prasad [16] introduced the Durrmeyer variant of the well-known Baskakov operators. However, there were some technical problems in the main estimates of [16], which were later improved by Sinha et al. [19]. In this continuation, in 1994, Gupta proposed yet another Durrmeyer type generalization of Baskakov operators by taking the weights of Beta basis function. The operators discussed in [8] provide better approximation in simultaneous approximation than the usual Baskakov–Durrmeyer operators, studied in [19]. This motivated us to study further in this direction and here, we propose the (p, q)-variant of Baskakov–Beta operators.