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Note on the weak–strong uniqueness criterion for the β-QG in Morrey–Campanato space

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ABSTRACT

Consider the β-quasi-geostrophic equations with the initial data $\theta_0 \in L^2(\mathbb{R}^2)$. Let θ and $\widetilde{\theta}$ be two weak solutions with the same initial value θ_0. If θ satisfies the energy inequality and if

$$\nabla\theta \in L^r 0, T; \dot{\mathcal{M}}_{p,q}(\mathbb{R}^2) \text{ with } \frac{2}{p}+\frac{\alpha}{r}=\alpha+\beta-1 \text{ and } \frac{2}{\alpha+\beta-1}<p<\frac{3}{r},$$

where $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)$ denotes the homogeneous Morrey–Campanato space, then we have $\theta=\widetilde{\theta}$. Due to the embedding relation $L^p(\mathbb{R}^2)\subset\dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)$ with $1<p<q<\infty$, we see that our result is an improvement of the corresponding result given by Zhao and Liu (2013).

1. Introduction

We consider the following two-dimensional β-generalized quasi-geostrophic equations (see e.g. [24,25]):

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \Lambda^\alpha\theta = 0,\\ \theta(x,0)=\theta_0(x),\end{cases} \tag{1.1}$$

where $\theta(x, t)$ is a scalar function representing the temperature, $u(x, t)$ is the velocity field of the fluid with $\nabla\cdot u = 0$ and determined by the Riesz transforms of the potential temperature θ:

$$u=\Lambda^{1-\beta}\mathcal{R}^{\perp}\theta=\Lambda^{1-\beta}(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \tag{1.2}$$

where $\Lambda=(-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and $\mathcal{R}_i, i=1,2$ are the usual Riesz transforms in $\mathbb{R}^2$. Here the Riesz potential operator $\Lambda^\alpha=(-\Delta)^{\frac{\alpha}{2}}$ is defined through the Fourier transform [20]:

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\widehat{f}(\xi).$$

where $\widehat{f}$ denotes the Fourier transform of f. Here $0<\alpha\le 2$ and $1\le\beta<2$ are two fixed parameters.

The β-generalized surface quasi-geostrophic Eqs. (1.1) and (1.2) was introduced by Kiselev [11] which is deeply related to the important model in geophysical fluid dynamics used in meteorology and oceanography, see [11,25] and the references therein.

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When $\beta = 1$, the surface quasi-geostrophic Eqs. (1.1) and (1.2) has been intensively investigated due to both its mathematical importance and its potential for applications as models of atmospheric and ocean fluid flow. For more details about its physical background, we can refer to Constantin et al. [2], Pedlosky [19] and references therein. The surface quasi-geostrophic equation with subcritical ($1 < \alpha \leq 2$) or critical dissipation ($\alpha = 1$) have been shown to possess global classical solutions whenever the initial data are sufficiently smooth. However, the global existence of smooth and uniqueness issue remains open for the supercritical case ($0 < \alpha < 1$). Various regularity (or blow-up) criteria have been produced to shed light on this difficult global regularity problem (see e.g. [1,3,4,12] and the references therein). This work is partially motivated by the recent progress on the 2D incompressible generalized MHD system, we refer to [7,28] and the references therein. The above works were subsequently extended to the case $1 \leq \beta < 2$ by [3,13,17,21–24,26,27].

However, the uniqueness of such weak solution for the 2D quasi-geostrophic equation is still open. When $1 < \alpha \leq 2$, Constantin and Wu [4] showed the following weak–strong uniqueness results: if a weak solution lies in the regular class

$$\theta \in L^r\big(0, T; L^p(\mathbb{R}^2)\big) \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} = \alpha - 1, \ \ 1 < r < \infty, \tag{1.3}$$

then there is at most one solution to the quasi-geostrophic equation with the initial data $\theta_0 \in L^2(\mathbb{R}^2)$. That is to say, all the weak solutions with the same initial data θ_0 coincide with the strong solution. Later on, Dong and Chen [5] more or less established the weak–strong uniqueness for the critical and supercritical dissipative quasi-geostrophic equations in the regularity class

$$\nabla\theta \in L^r\big(0, T; L^p(\mathbb{R}^2)\big) \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} = \alpha, \quad \frac{2}{\alpha} < r < \infty.$$

Recently, Dong and Chen [6] have refined the above conditions to the following uniqueness criterion in the framework of Besov spaces: the weak–strong solutions are unique in the class

$$\nabla\theta \in L^r\big(0, T; B^0_{p,\infty}(\mathbb{R}^2)\big) \quad \text{for} \quad \frac{2}{p} + \frac{\alpha}{r} = \alpha, \quad \frac{2}{\alpha} < p < \infty \text{ and } 0 < \alpha \leq 2.$$

In [16], Marchand has proved the weak–strong uniqueness for the critical case ($\alpha = 1$) when one of the two weak solutions θ satisfies

$$\theta \in L^\infty(0, T; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^2)) \cap L^2\big(0, T; L^2(\mathbb{R}^2)\big) \cap L^\infty(0, T; \mathrm{BMO}(\mathbb{R}^2))$$

and the norm $\|\theta\|_{L^\infty(0,T;\mathrm{BMO}(\mathbb{R}^2))}$ is small enough, where BMO is the space of bounded mean oscillations. Recently, for $0 < \alpha < 2$, Liu et al. [15] have refined the above result in the following sense

$$\nabla\theta \in L^1\big(0, T; \mathrm{BMO}(\mathbb{R}^2)\big).$$

It should be mentioned that the Serrin's (1.3) condition implies a connection of regularity criteria of weak solutions and a scaling invariance property, that is, solves (1.1) and (1.2) if and only if

$$\theta_\lambda(x, t) = \lambda^{\alpha+\beta-2}\theta(\lambda^\alpha t, \lambda x)$$

solves (1.3) and the scaling invariance

$$\|\theta\|_{L^r((0,\lambda^\alpha T);L^p)} = \|\theta_\lambda\|_{L^r((0,T);L^p)}$$

holds true for all $\lambda > 0$ if and only if r and p satisfy the Serrin's condition (1.3). Due to its similarity with 3D incompressible Navier–Stokes/Euler equations, we consider the case $2 < \alpha + \beta < 3$ as the subcritical case, the case $\alpha + \beta = 2$ as the critical case, and the case $1 < \alpha + \beta < 2$ as the supercritical case. While the regularity and uniqueness of global weak solutions of the Navier–Stokes/Euler equations remains an outstanding open problem in mathematical physics, the global well-posedness of Eqs. (1.1) and (1.2) seems to be in a satisfactory situation in the subcritical and critical cases. Recently, the global existence of smooth solutions of Eqs. (1.1) and (1.2) with suitable choices of α and β has been studied by many authors (cf. [22]), the uniqueness of weak solution in the critical case and supercritical case is a rather challenging problem in mathematical fluid mechanics.

The aim of this paper to improve and extend the above results in the following sense that if θ is a weak solution toEqs. (1.1) and (1.2) and if

$$\nabla\theta \in L^r\Big(0, T; \dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)\Big) \quad \text{with} \quad \frac{2}{p} + \frac{\alpha}{r} = \alpha + \beta - 1, \quad \frac{2}{\alpha + \beta - 1} < p < \infty,$$

then, θ is unique in the class of weak solutions. Here $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)$ stands for the homogeneous Morrey–Campanato space. We point out here that the Morrey–Campanato spaces have been studied by Lemarié–Rieusset and co-workers [14]. They are useful tools for stating minimal regularity requirements on the coefficients of partial differential operators for proving regularity or uniqueness of solutions. Since the following embedding relation $L^p(\mathbb{R}^2) \subset \dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)$ with $1 < q < p < \infty$ holds, our uniqueness criterion can be understood as an extension of the uniqueness result of Zhao–Liu [25]. It is a natural way to extend the space widely and improve the previous results. However, the spaces $\dot{\mathcal{M}}_{p,q}(\mathbb{R}^2)$ and $B^0_{p,\infty}(\mathbb{R}^2)$ are different and no inclusion relation between them.