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Picard iteration and Padé approximations for stiff fractional point kinetics equations

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ABSTRACT

A model of stiff point kinetics equations is one of the important models in the nuclear reactor dynamics. This model describes the neutron density and the precursor concentrations of delayed neutrons into nuclear reactors. In this work, a fractional model of the stiff point kinetics equations is studied to describe the neutron density behavior by the fractional order. Picard iteration and Padé approximations are presented to solve the stiff fractional point kinetics equations with multi-group of delayed neutrons. The validity of the fractional model is tested for different cases of step, ramp and sinusoidal reactivities. The numerical results of Picard iteration and Padé approximations are computed for various fractional order. The results of Padé11 approximation are in good agreement with the results of Picard iteration than Padé01 approximation. In addition, the numerical results confirm that the neutron density for a positive (negative) reactivity is increasing (decreasing) quicker with decreases the fractional order.

1. Introduction

In recent years, the scientists used fractional calculus to describe the physical phenomena. The fractional calculus is the study of the theory of the integrals and derivatives of arbitrary order. It is more realistic and more popular. Also, the ordinary differential operator is a local operator but the fractional order differential operator is non-local [1,2]. In addition, the fractional models play an outstanding role from the viewpoint of applications in modeling of many reaction processes, biomathematical, economics and many problems in chemistry, physics and engineering [3,4]. One of their applications is the nuclear reactor dynamics. The point reactor kinetics model represents one of the important models in the nuclear reactor dynamics. This model is the subject of countless studies and applications to understand the neutron dynamics and its effects. This model describes the neutron density and the precursor concentration of delayed neutrons into the nuclear reactors. The neutron density is important to calculate the power of reactor and to study the safety of the nuclear reactors. For this reason, the interest of the nuclear reactor scientists to study the fractional neutron point kinetics equations (FNPKE) increased. The first scientific group which derived the FNPKE is Espinosa-Paredes et al. [5]. In 2014, Ray and Patra [6] presented numerical simulation for solving another formula of fractional point kinetic equations using the multi-step differential transform method. The main goal of this work is to study the fractional model [6] for point kinetics equations using different approximation methods. These methods are Picard iterative method [7–10], Padé01 (Backward Euler), and Padé11 (Crank–Nicholson) approximations [11–17]. Whereas, Picard iteration is a constructive procedure for establishing the

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existence and uniqueness of solutions of first order systems of differential equations [7]. While, Padé approximations are widely used to solve the systems of first order differential equations. In addition, the scheme of Padé01 and Padé11 approximations are A-stable [14] or unconditionally stable in view of the matrix analysis method [16].

This work is organized as follows. The basic definitions of the fractional calculus are introduced in Section 2. The solutions of the matrix form of the fractional differential equation based on Picard iterative method, Backward Euler and Crank–Nicholson approximations are presented in Section 3. These methods are applied to solve the fractional point kinetics equations with multi-group of delayed neutrons in Section 4. The numerical results of these methods are discussed in Section 5. The conclusion with a brief summary of the main findings is reported in Section 6.

2. Basic definitions of fractional calculus

In this section, we mention the following basic definitions of fractional calculus which are used further in the present work.

Definition 1 [18]. The Riemann–Liouvill fractional integral operator of order $\kappa > 0$, of a function $f(t) \in C_\mu$ and $\mu \geq -1$ is defined as follows:

$$I_t^\kappa f(t) = \begin{cases} \dfrac{1}{\Gamma(\kappa)} \displaystyle\int_0^t (t-\tau)^{\kappa-1} f(\tau) d\tau, & \kappa > 0; \\ f(t), & \kappa = 0. \end{cases} \tag{1}$$

According to the Riemann–Liouvill fractional integral, we have

$$I_t^\kappa t^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\kappa)} t^{\nu+\kappa}. \tag{2}$$

Definition 2 [19]. The fractional derivative of $f(t)$ in the Caputo sense is defined as follows:

$$D_t^\kappa f(t) = I_t^{m-\kappa} D_t^m f(t) = \frac{1}{\Gamma(m-\kappa)} \int_0^t (t-\tau)^{m-\kappa-1} f^{(m)}(\tau) d\tau, \tag{3}$$

where m is integer number and $m-1 < \kappa \leq m, t > 0$. Based on the Caputo definition (3), the following results are obtained

$$D_t^\kappa t^\nu = \begin{cases} \dfrac{\Gamma(\nu+1)}{\Gamma(\nu+1-\kappa)} t^{\nu-\kappa}, & \nu = 1, 2, \ldots; \\ 0, & \nu = 0. \end{cases} \tag{4}$$

For the Riemann–Liouvill fractional integral and Caputo fractional derivative, we get

$$I_t^\kappa D_t^\kappa f(t) = f(t) - \sum_{r=0}^{m-1} f^{(r)}(0_+) \frac{t^r}{r!}, \quad m-1 < \kappa \leq m. \tag{5}$$

3. Solution of fractional differential equations

We use three different methods, Picard iteration, Padé01 and Padé11 approximations, to solve the system of fractional differential equations.

3.1. Picard iterative method

To illustrate the basic idea of Picard iterative method [7–9], the system of the fractional differential equations of an arbitrary order $\kappa > 0$ is considered as

$$D_t^\kappa \mathbf{X}(t) = \mathbf{F}(t, \mathbf{X}(t)), \quad \mathbf{X}(0) = \mathbf{X}_0, \tag{6}$$

where D_t^κ is the fractional differential operator of order $m-1 < \kappa \leq m$, m is integer number, and $\mathbf{X}(t)$ is the vector of unknown variables.

The required solution, $\mathbf{X}(t)$, for Eq. (6) can be obtained as the limit of a sequence of functions $\mathbf{X}_{r+1}(t)$ generated by the recurrence formula

$$\mathbf{X}_{r+1}(t) = \mathbf{X}_0 + I_t^\kappa \mathbf{F}(t, \mathbf{X}_r(t)), \quad r = 0, 1, 2, 3, \ldots \tag{7}$$

where $\mathbf{X}_0$ is the initial value of the vector of unknown variables, and $\mathbf{X}(t) = \lim_{r\to\infty} \mathbf{X}_r(t)$.

Let us assume that the function $\mathbf{F}(t, \mathbf{X}(t))$ is a linear function of the vector of the unknown variables

$$D_t^\kappa \mathbf{X}(t) = \mathbf{F}(t, \mathbf{X}(t)) = \mathbf{A}\mathbf{X}(t), \tag{8}$$

where $\mathbf{A}$ is a constant matrix during the small time interval $[0, t]$.