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Solving differential equations of fractional order using an optimization technique based on training artificial neural network

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ABSTRACT

The current study aims to approximate the solution of fractional differential equations (FDEs) by using the fundamental properties of artificial neural networks (ANNs) for function approximation. In the first step, we derive an approximate solution of fractional differential equation (FDE) by using ANNs. In the second step, an optimization approach is exploited to adjust the weights of ANNs such that the approximated solution satisfies the FDE. Different types of FDEs including linear and nonlinear terms are solved to illustrate the ability of the method. In addition, the present scheme is compared with the analytical solution and a number of existing numerical techniques to show the efficiency of ANNs with high accuracy, fast convergence and low use of memory for solving the FDEs.

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1. Introduction

In recent years, the fractional calculus has gained a great development in both theory and application. Its appearance and development, to a certain extent, make up for defects of the classical calculus of integer order. The fractional calculus has been used to describe many phenomenons in almost all applied sciences, such as fluid flow in porous materials, anomalous diffusion transport, acoustic wave propagation in viscoelastic materials, signal processing, nanotechnology, financial theory, electric conductance of biological systems and others (see, [5–9,44]).

Fractional differential equation (FDE) is a unifying theory expressing memory and hereditary characteristics of diverse materials and process in comparison with classical integer order derivative [8,24,30]. However, the progress of numerical techniques in this area has received huge attention and has undergone a rapid growth in recent years. These methods include spectral methods [3,4,10,11,12,14,15,22], fractional linear multi-step methods [19,43], Adomian decomposition method [31,32], Wavelet Method [23], Laplace transforms [40,41] and variational iteration method [35].

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Establishing the aforementioned demands, we aims to illustrate the solution behavior of the artificial neural networks (ANNs) approximations for multi-term fractional differential equations (FDEs) of the form

$$\left({}_a^C D_t^\alpha x\right)(t) = f(t, x(t)), \quad a \le t \le b, \tag{1.1}$$

provided that the initial conditions

$$y^{(i)}(0) = \gamma_i, \quad i = 0, \ldots, n-1, \tag{1.2}$$

where $m-1 < \alpha \le m$, $m \in \mathbb{N} := \{1, 2, \ldots\}$, $\gamma_i \in \mathbb{R} := (-\infty, \infty)$, ${}_a^C D_t^\alpha$ is the Caputo-type derivative and $f : [0, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. To interpret the performance and behavior of a fractional dynamic systems, it is essential to exploit a suitable definition of the derivative of fractional order. We have several types of fractional derivatives such as Riemann–Liouville, Caputo, Grünwald–Letnikov, Riesz and Weyl that all these fractional derivative order definitions have their advantages and disadvantages. Among these fractional derivatives, the Riemann–Liouville and Caputo derivatives are the most common and popular fractional derivatives. However, the Riemann–Liouville derivative has certain disadvantages when trying to model real-world phenomena with FDEs. The Riemann–Liouville derivative of a constant is not zero. In addition, if an arbitrary function is a constant at the origin, its fractional derivation has a singularity at the origin for instant exponential and Mittag–Leffler functions. Theses disadvantages reduce the field of application of the Riemann–Liouville fractional derivative. Besides that, the main advantage of the Caputo's approach, which makes it more popular is that the initial conditions for the FDEs with the Caputo takes on the same form as for integer-order differential equations [20,21,36,48]. Although the structure of the approximate solution based on the present method is such that there is no requirement to have constraints for initial conditions satisfaction, in order to design the artificial neural network (ANN), it is more reasonable to use such type of differentiability.

In recent years, ANNs were attained a considerable attention as robust and effective tools for function approximation [1,2,25,33]. It is proved that a multi-layer perceptron can approximate continuous functions over a compact subset of R^n. Cybenko [13] presented a theorem to demonstrate the capabilities of the ANNs-based function approximation for the sigmoid activation functions. In a wide range of mathematical problems, we look for an unknown continuous function, satisfying some conditions. For example, in ordinary and partial differential equation theory as well as in the theory of integral equations and optimal control theory. For that reason, the work of Lagaris et. al [27] has an important rule in literature for approximating the solution of ordinary and partial differential equations. The application of ANNs for solving differential equations was not limited to ordinary and partial differential equations. Effati and Pakdaman [16] used the ability of ANNs-based function approximation, for solving fuzzy differential equations. They also in Effati and Pakdaman [17] used the ANNs methodology to approximate the state, co-state and control functions for an optimal control problem. Effati and Buzhabadi [18] used feed-forward ANNs to approximate the solution of Fredholm integral equations of the second kind. Using a ANN methodology can offer a differentiable continuous solution of differential equation, which satisfies all initial or boundary conditions, while the accuracy of the method can be adjusted by changing the parameters of ANN. Hence, Sabouri et al. [39] employed the ability of neural networks for solving fractional order optimal control problems. Jafariana et al. [26] presented an ANN approach for solving a class of FDEs. They used an unsupervised back-propagation learning algorithm for adjusting the weights of ANN and a suitable truncated power series of the solution function. In the current approach, we solve the FDEs directly using an unconstrained optimization problem which can be solved by any optimization technique. The trial solutions of the proposed ANN involve a single independent variable regardless of the dimension of the problem. The approximate solutions are continuous over all the domain of integration.

Recently, Raja et al. [45,46] presented a stochastic computational intelligence approach based on the strength of feed forward ANNs and Genetic algorithm for the solution of FDEs. However, for nonlinear FDEs, the accuracy is low. Thereafter, Raja et al. [47] proposed ANNs and sequential quadratic programming for solving fractional order Riccati equations. However, performance of perceptron ANNs, motivated the authors to use them to approximate the solution of FDEs. One of important advantages of the present ANNs for the function approximation is that they propose the solution of differential equations as a differentiable function. Although the ANN trains at a number of points, the solution can be calculated at each arbitrary point in training interval even between training points. To the best of authors' knowledge, such approach for solving the FDEs has not been reported so far in the literature which partially motivated us to propose this efficient and applicable technique for the solution of multi-terms FDEs.

To summarize, solving FDEs with the corresponding trained ANN technique offers the following preferences in comparison with the classical numerical schemes:

1. Solution search proceeds without coordinate transformations.
2. ANN learns to solve the DE analytically.
3. Computational complexity does not increase quickly when the number of sampling points increase.
4. Rapid calculation of the solution values.

Contrary to the above advantages, standard numerical methods such as predictor-correction method [49], Runge–Kutta method [51], fractional Euler method [34] and operational matrix methods [22,38] require the discretization of domain into the number of finite domains/points where the functions are approximated locally and their computational complexity increases rapidly with the number of sampling points [37,42]. Besides that, rounding-off errors solemnly influence the solution precision in the numerical techniques with complicatedness that also raise quickly with the number of sampling points [37].

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