Applied Mathematics and Computation 293 (2017) 218–225

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some results on the distance and distance signless Laplacian spectral radius of graphs and digraphs[☆]

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ARTICLE INFO

Keywords:
Distance matrix
Distance signless Laplacian matrix
Dichromatic number
Spectral radius

ABSTRACT

Let $\rho(D(G))$ denote the distance spectral radius of a graph G and $\partial(\overrightarrow{G})$ denote the distance signless Laplacian spectral radius of a digraph $\overrightarrow{G}$. Let $\mathcal{G}_{n,k}^{D}$ be the set of all k-connected graphs of order n with diameter D. In this paper, we first determine the unique graph with minimum distance spectral radius in $\mathcal{G}_{n,k}^{D}$; we then give sharp upper and lower bounds for the distance signless Laplacian spectral radius of strongly connected digraphs; we also characterize the digraphs having the maximal and minimal distance signless Laplacian spectral radii among all strongly connected digraphs; furthermore, we determine the extremal digraph with the minimal distance signless Laplacian spectral radius with given dichromatic number.

1. Introduction

In this paper we consider simple and connected graphs. Let $A(G) = (a_{ij})_{n\times n}$ be the (0, 1)-adjacency matrix of G, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$ otherwise. The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. The distance between vertices v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path from v_i to v_j. The diameter of a graph G, denoted by D, is the maximum distance between any two vertices of G. The distance matrix of G, denoted by $D(G)$, is the symmetric real matrix with (i,j)-entry being $d_G(v_i, v_j)$ (or d_{ij}). The distance eigenvalues (resp. distance spectrum) of G, denoted by

$$\lambda_1(D(G)) \geq \lambda_2(D(G)) \geq \cdots \geq \lambda_n(D(G)).$$

In particular, $\lambda_1(D(G))$, as the largest eigenvalue of $D(G)$, will be called the distance spectral radius (or index) of $D(G)$, denoted by $\rho(D(G))$.

Let $\overrightarrow{G} = (V(\overrightarrow{G}), E(\overrightarrow{G}))$ be a digraph, where $V(\overrightarrow{G})$ and $E(\overrightarrow{G})$ are the vertex set and arc set of $\overrightarrow{G}$, respectively. A digraph $\overrightarrow{G}$ is *strongly connected* if for every pair of vertices $x, y \in V(\overrightarrow{G})$ there exists a directed path from x to y.

Let $\overrightarrow{G}$ be a finite, simple strongly connected digraph. Two vertices are called *adjacent* if they are connected by an arc. If $e = uv \in E(\overrightarrow{G})$, then u is the *initial* vertex of e and v is the *terminal* vertex of e. For a vertex $v \in V(\overrightarrow{G})$, $N^{+}_{\overrightarrow{G}}(v)$ and $N^{-}_{\overrightarrow{G}}(v)$ denote the set of out-neighbors and in-neighbors of v, respectively. Let $d^{+}_{\overrightarrow{G}}(v) = |N^{+}_{\overrightarrow{G}}(v)|$ and $d^{-}_{\overrightarrow{G}}(v) = |N^{-}_{\overrightarrow{G}}(v)|$ denote the

☆ Supported by NSFC (nos. 11531011 and 11461071), NSFXJ (no. 2015KL019).
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http://dx.doi.org/10.1016/j.amc.2016.08.025
0096-3003/

out-degree and in-degree of v in $\overrightarrow{G}$, respectively. In particular, let δ^- denote the minimum in-degree of $\overrightarrow{G}$. Let $A(\overrightarrow{G})$ denote the adjacency matrix of $\overrightarrow{G}$ whose entry a_{ij} is defined as $a_{ij} = 1$ if $v_iv_j \in E(\overrightarrow{G})$ and $a_{ij} = 0$ otherwise.

Let $d_{ij} = d_{\overrightarrow{G}}(v_i, v_j)$ be the length of shortest dipath from v_i to v_j in $\overrightarrow{G}$. We call $D_i = \sum_{j=1}^{n} d_{ij}$ the *distance degree* of the vertex v_i $(i = 1, 2, \ldots, n)$. Let $D(\overrightarrow{G}) = (d_{ij})$ and $D^Q(\overrightarrow{G}) = Diag(Tr) + D(\overrightarrow{G})$ denote the *distance matrix* and *distance signless Laplacian matrix* of $\overrightarrow{G}$ respectively, where $Diag(Tr)$ is the diagonal matrix with D_i. Clearly, the matrix $D^Q(\overrightarrow{G})$ is irreducible when $\overrightarrow{G}$ is strongly connected. The eigenvalue of $D^Q(\overrightarrow{G})$ with the largest modulus is the *distance signless Laplacian spectral radius* of $\overrightarrow{G}$, denoted by $\partial(\overrightarrow{G})$.

The distance matrix of a connected graph is very useful in different fields including the design of communication networks, graph embedding theory as well as molecular stability and has been studied extensively. Please refer to [7,10–16] for previous results about distance spectrum of graphs. In [2], Balaban et al. proposed the use of a distance spectral radius as a molecular descriptor, and in [4], a distance spectral radius was successfully used to infer the extent of branching and to model boiling points of an alkane. Therefore, the study concerning the maximal (minimal) distance spectral radius of a given class of graphs is of great interest and significance. Zhang and Godsil [19] determined the graph with k cut vertices (respectively, k cut edges) with the minimal distance spectral radius. Furthermore, Zhang [20] determined the unique graph with minimum distance spectral radius among all connected graphs of order n with a given diameter. Lin et al. [8] characterized the extremal digraphs with minimum distance spectral radius among all digraphs with given vertex connectivity and the extremal graphs with minimum distance spectral radius among all graphs with given edge connectivity. Moreover, they gave the exact value of the distance spectral radius of those extremal digraphs and graphs. Moreover, they also characterized the graphs with the maximum distance spectral radius among all graphs of fixed order with given vertex connectivity 1 and 2. Lin and Shu [9] characterized the digraphs having the maximal and minimal distance spectral radii among all strongly connected digraphs, and determined the extremal digraph with the minimal distance spectral radius with given arc connectivity and dichromatic number. Hansen and Stevanović [5] determined the graphs with maximum spectral radius among all connected graphs of order n with diameter D. Huang et al. [6] generalized this result to k-connected graphs of order n with diameter D. Aouchiche and Hansen [1] defined the distance Laplacian and the distance signless Laplacian of a connected graph.

Let $\mathcal{G}_{n,k}^D$ be the set of all k-connected graphs of order n with diameter D. The current paper is organized as follows. In Section 2, we determine the unique graph with minimum distance spectral radius in $\mathcal{G}_{n,k}^D$. In Section 3, we give sharp upper and lower bounds for the distance signless Laplacian spectral radius of digraphs. In Section 4, we characterize maximal and minimal distance signless Laplacian spectral radius for digraphs. In Section 5, we determine the extremal digraph with minimal distance signless Laplacian spectral radius with given dichromatic number.

2. The minimal distance spectral radius of *k*-connected graphs with given diameter

The *sequential join* $G_1 \vee G_2 \vee \cdots \vee G_k$ of graphs $G_1, G_2, \ldots, G_k$ is the graph formed by taking one copy of each graph and adding additional edges from each vertex of G_i to all vertices of G_{i+1}, for $1 \le i \le k-1$. The diameter of a graph G may decrease (or remain unchanged) when a new edge e is added to it. We say that G is *diameter critical* when the addition of any edge decreases the diameter. The structure of a diameter critical graph has been obtained by Ore [18].

Lemma 2.1 [18]**.** *A graph G with diameter D is diameter critical if and only if G is isomorphic to* $K_1 \vee K_{n_1} \vee \cdots \vee K_{n_{D-1}} \vee K_1$.

Lemma 2.2 [18]**.** *A k connected graph G with diameter D is diameter critical if and only if G is isomorphic to* $K_1 \vee K_{n_1} \vee \cdots \vee K_{n_{D-1}} \vee K_1$ *with* $n_i \ge k$, $1 \le i \le D-1$.

Let $\mathcal{G}_{n,k}^D$ be the set of all k-connected graphs of order n with diameter D. In this section, we determine the unique graph with minimum distance spectral radius in $\mathcal{G}_{n,k}^D$.

The following lemma is an immediate consequence of the Perron–Frobenius theorem.

Lemma 2.3. *Let G be a connected graph with* $u, v \in V(G)$ *and* $uv \notin E(G)$. *Then* $\lambda_1(D(G)) > \lambda_1(D(G+uv))$.

Let G^* be the graph which has the minimum distance spectral radius in $\mathcal{G}_{n,k}^D$. By Lemma 2.3, we know that G^* must be a diameter critical graph since we can decrease the distance spectral radius by adding edges. Thus G^* has the structure presented by Lemma 2.2. Let $V_i = V(K_{n_i})$, then $|V_i| = n_i \ge k$ for $1 \le i \le D-1$. Suppose $\mathbf{x}(G^*)$ is the Perron vector corresponding to $\lambda_1(D(G^*))$. If vertices u and v belong to the same V_i, $1 \le i \le D-1$, which means u and v are similar vertices, then the components of $\mathbf{x}(G^*)$ corresponding to u and v are equal. Thus, we can denote x_i the components of $\mathbf{x}(G^*)$ corresponding to the vertices in V_i, $i \in \{1, \ldots, D-1\}$.

Lemma 2.4 [17]**.** *If A is a nonnegative and irreducible* $n \times n$ *matrix with largest eigenvalue* $\rho(A)$ *and row sums* $s_1, s_2, \ldots, s_n$, *then*

$$min_{1\le i\le n} s_i \le \rho(A) \le max_{1\le i\le n} s_i$$

Moreover, one of the equalities holds if and only if row sums of A are equal.