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Singularity of Hermitian (quasi-)Laplacian matrix of mixed graphs

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ABSTRACT

A mixed graph is obtained from an undirected graph by orienting a subset of its edges. The Hermitian adjacency matrix of a mixed graph M of order n is an $n \times n$ matrix $H(M) = (h_{kl})$, where $h_{kl} = -h_{lk} = i$ $(i = \sqrt{-1})$ if there exists an orientation from v_k to v_l and $h_{kl} = h_{lk} = 1$ if there exists an edge between v_k and v_l but not exist any orientation, and $h_{kl} = 0$ otherwise. Let $D(M) = diag(d_1, d_2, \dots, d_n)$ be a diagonal matrix where d_i is the degree of vertex v_i in the underlying graph M_u. Hermitian matrices $L(M) = D(M) - H(M)$, $Q(M) = D(M) + H(M)$ are said as the Hermitian Laplacian matrix, Hermitian quasi-Laplacian matrix of mixed graph M, respectively. In this paper, it is shown that they are positive semi-definite. Moreover, we characterize the singularity of them. In addition, an expression of the determinant of the Hermitian (quasi-)Laplacian matrix is obtained.

1. Introduction

All graphs considered in this paper are simple and connected. Let G be an undirected graph of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The adjacency matrix $A(G) = (a_{ij})_{n\times n}$ of G is defined as follows: $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. A *mixed graph* M is obtained from an undirected graph G by orienting a subset of its edges. We call G the *underlying graph* of M, denoted by M_u. The vertex set of M is denoted by $V(M)$, which is the same as $V(G)$. The edge set $E(M)$ is the union of the set of undirected edges $E_0(M)$ and the set of directed edges (arcs) $E_1(M)$. We distinguish undirected edge as $x\leftrightarrow y$ of vertices, while the directed edge (arcs) is $x \to y$ if the orientation is from x to y. If not considering the direction, we write xy to be an edge of M.

The *Hermitian adjacency matrix* of a mixed graph M of order n is an $n \times n$ matrix $H(M) = (h_{kl})$, where $h_{kl} = -h_{lk} = i$ $(i = \sqrt{-1})$ if there exists an orientation from v_k to v_l and $h_{kl} = h_{lk} = 1$ if there exists an edge between v_k and v_l but not exist any orientation, and $h_{kl} = 0$ otherwise. That is

$$h_{kl} = \begin{cases} 1 & \text{if } v_k \leftrightarrow v_l, \\ i & \text{if } v_k \to v_l, \\ -i & \text{if } v_k \leftarrow v_l, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously this matrix is Hermitian and all eigenvalues are real. It was introduced by Liu and Li [25] and independently by Guo and Mohar [20].

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Two present authors gave the definition of Hermitian Laplacian matrix of mixed graph [29]. The Hermitian Laplacian matrix $L(M)$ of a mixed graph M is defined as $D(M) - H(M)$ where $D(M) = diag(d_1, d_2, \dots, d_n)$ is a diagonal matrix where d_i is the degree of vertex v_i in the underlying graph M_u [29]. In [29], it was shown that Hermitian Laplacian matrix is positive semi-definite. Moreover, some properties for its spectrum were investigated. As shown in [13], signless Laplacian matrix of undirected graph seems to be the most convenient for use in studying graph properties and attracted some attentions of researchers. Together with [10–12], there are about 100 papers on the signless Laplacian matrix of undirected graph published since 2005. Motivated from this, we introduce the Hermitian quasi-Laplaician matrix $Q(M) = D(M) + H(M)$ of mixed graph M. It is evident that $Q(M)$ is Hermitian and all eigenvalues are real.

An $i_1 - i_k$-walk W in a mixed M is a sequence $W : v_{i_1} v_{i_2} \dots v_{i_k}$ of vertices such that for $1 \le s \le k-1$ we have $v_{i_s} \leftrightarrow v_{v_{s+1}}$ or $v_{i_s} \to v_{v_{s+1}}$ or $v_{i_s} \leftarrow v_{v_{s+1}}$. An $i_1 - i_k$-walk W is called *even* (*odd*) if k is even (odd). The value of a mixed walk $W = v_1 v_2 v_3 \dots v_l$ is $h(W) = h_{12} h_{23} \dots h_{(l-1)l}$. A mixed walk is *positive* or *negative* if $h(W) = 1$ or $h(W) = -1$, respectively. A mixed is said *imaginary* if $h(W) = \pm i$. Note that for one direction the value of a mixed walk or a mixed cycle is α, then for the reversed direction its value is $\overline{\alpha}$. Thus, if the value of a mixed cycle is 1 (resp. -1) in a direction, then its value is 1 (resp. -1) for the reversed direction. In these situations, we just termed this mixed cycle as a positive (negative) mixed cycle without mentioning any direction. A graph is *positive* (resp. *negative*) if each its mixed cycle is positive (resp. negative). An induced subgraph of M is an induced subgraph of its underlying graph G with the same orientations. For a subgraph H of M, let $M - H$ be the subgraph obtained from M by deleting all vertices of H and all incident edges. For $V_1 \subseteq V(M)$, $M - V_1$ is the subgraph obtained from M by deleting all vertices in V_1 and all their incident edges.

Up to now, there exist considerably fewer results on the Hermitian spectra of mixed graph (see [7,20,25,26,29]). We organize this paper as follows. In Section 2, we characterize the singularity of Hermitian Laplacian matrix of mixed graphs. In addition, we derive an expression of determinant of Hermitian Laplacian matrix. In Section 3, we characterize the singularity of Hermitian quasi-Laplacian of mixed graphs. Moreover, an expression of determinant of Hermitian quasi-Laplacian are present.

2. Hermitian Laplacian matrix of mixed graphs

Let $S(M) = (s_{ke})$ be an $n \times m$ matrix indexed by the vertex and the edge of mixed graph M, where s_{ke} is a complex number and $|s_{ke}| = 1$ or 0. Moreover,

$$s_{ke} = \begin{cases} -s_{le}, & \text{if } v_k \leftrightarrow v_l \\ -i \cdot s_{le}, & \text{if } v_k \to v_l \\ i \cdot s_{le}, & \text{if } v_k \leftarrow v_l \\ 0, & \text{otherwise.} \end{cases}$$

It is evident that $S(M)$ is not unique. A matrix $S(M)$ is said an *incidence matrix* of M.

Theorem 1. [29] *Let M be a mixed graph. Then $L(M) = S(M)S^*(M)$ and $L(M)$ is a positive semi-definite matrix.*

Theorem 2. *Let M be a connected mixed graph on vertices $v_1, v_2, \dots, v_n$. Then $L(M)$ is singular if and only if any $1 - i$-walk has the same value. In this case, 0 is a simple eigenvalue with an eigenvector $\alpha = \left(1, \overline{h(W_2)}, \overline{h(W_3)}, \dots, \overline{h(W_n)}\right)^T$, where W_i is a $1 - i$-walk in M.*

Proof. Let $x^T = (x_1, x_2, \dots, x_n)$. Note that for an non-zero vector x, we have $L(M)x = 0$ if and only if $S^*(M)x = 0$. By $(S^*(M)x)_e = \overline{s_{ke}} x_k + \overline{s_{le}} x_l$, we have $S^*(M)x = 0$ if and only if $x_k = h_{kl} x_l$ for any edge $e = v_k v_l$. Let $W = u_1 u_2 \dots u_i$ be any $1 - i$-walk and $u_1 = v_1$, $u_i = v_i$. So we have

$$x_1 = h_{12} x_2 = h_{12} h_{23} x_3 = \dots = h_{12} h_{23} \dots h_{i-1,i} x_i = h(W) x_i,$$

this implies that each $1 - i$-walk has the same value. So we have

$$\begin{aligned} x^T &= (x_1, x_2, \dots, x_n) \\ &= (x_1, \overline{h(W_2)} x_1, \overline{h(W_3)} x_1, \dots, \overline{h(W_n)} x_1) \\ &= x_1 \alpha^T. \end{aligned}$$

This implies that 0 is a simple eigenvalue of $L(M)$ with an eigenvector α.

Conversely, let $\alpha = (a_1, a_2, \dots, a_n)^T$ be a vector. Note that $a_k = h_{kl} a_l$ for each edge $e = v_k v_l$. So, we have

$$\begin{aligned} \alpha^* L(M) \alpha &= \sum_{e \in E(M)} |a_k - h_{kl} a_l|^2 \\ &= \sum_{e \in E(M)} |h_{kl} a_l - h_{kl} a_l|^2 = 0. \end{aligned}$$

This implies that $L(M)$ is singular. $\square$