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A cubic trigonometric B-spline collocation approach for the fractional sub-diffusion equations

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ABSTRACT

A cubic trigonometric B-spline collocation approach for the numerical solution of fractional sub-diffusion equation is presented in this paper. The approach is based on the usual finite difference scheme to discretize the time derivative while the approximation of the second-order derivative with respect to space is obtained by the cubic trigonometric B-spline functions with the help of Grünwald–Letnikov discretization of the Riemann–Liouville derivative. The scheme is shown to be stable using the Fourier method and the accuracy of the scheme is tested by application to a test problem. The results of the numerical test verify the accuracy and efficiency of the proposed algorithm.

1. Introduction

1.1. Problem

The model problem considered is the one dimensional fractional sub-diffusion equation describing the sub-diffusive phenomena [16]

$$\frac{\partial}{\partial t}u(x,t) = K\,{}_0D_t^{1-\gamma}\frac{\partial^2}{\partial x^2}u(x,t) \qquad 0<\gamma<1,\;\; 0\le x\le L,\;\; t\ge 0 \tag{1.1}$$

with initial condition

$$u(x,0) = f(x) \qquad 0\le x\le 1 \tag{1.2}$$

and boundary conditions

$$u(0,t) = g_1(t),\, u(L,t) = g_2(t) \qquad t\ge 0 \tag{1.3}$$

where $K = 1$ is the diffusivity coefficient, $\gamma \in (0, 1)$ is the anomalous diffusion exponent and ${}_0D_t^{1-\gamma}$ denotes the Riemann–Liouville fractional derivative of order $1-\gamma$ for the function $\phi(x, t)$, defined in [1] as

$${}_0D_t^{1-\gamma}\phi(x,t) = \frac{1}{\Gamma(\gamma)}\frac{\partial}{\partial t}\int_0^t \frac{\phi(x,\tau)}{(t-\tau)^{1-\gamma}}d\tau. \tag{1.4}$$

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1.2. Applications

Recently, much attention in the literature has focused on the solution of fractional differential equations. There are number of phenomena in science and engineering which can be accurately modeled by fractional derivatives. Since, in certain circumstances, the description of important processes by means of differential equations involving fractional derivatives is more reliable and accurate, their numerical solutions have been the subject of much interest. The applications are wide and include viscoplastic and viscoelastic flow [2], control theory [3], transport problems [4], tumor development [5], random walks [6,7], continuum mechanics [8] and turbulence [9,10]. Due to diverse and increasing applications of fractional differential equations, it is important to explore efficient methods to find approximate solutions of fractional differential equations.

1.3. Theoretical aspects and literature review

Some numerical methods for solving fractional differential equations have been developed. Grünwald–Letnikov approach is used to discretize the fractional derivatives. A shifted Grünwald formula was proposed by Meerschaert and Tadjeran [11] for the approximation of space fractional derivatives of order $0 < \gamma < 1$. Yuste [12] has obtained numerical solutions of fractional diffusion equation by a weighted average finite difference method and has provided examples in which numerical results are compared against the exact solutions. An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equation is presented in [13] by Yuste and Acedo. Langlands and Henry [14] have investigated the accuracy and stability of the numerical solution of the fractional diffusion equation. Murio [15] have presented an implicit finite difference approximation for time fractional diffusion equations. Tasbozan et al. [16] have solved fractional diffusion equation for force free case using cubic B-spline collocation method. Gao and Sun [22] have derived a compact finite difference scheme for solving the fractional sub-diffusion equations. The authors applied an L_1 discretization for the time-fractional part and fourth-order accurate compact approximation for the second-order space derivative. Cui [23] has developed the high-order compact finite difference scheme for solving one-dimensional fractional diffusion equation which is based on Grünwald–Letnikov discretization of the Riemann–Liouville derivative to obtain a fully discrete implicit scheme.

In this paper, an approximation technique using trigonometric cubic B-splines is presented for the numerical solution of fractional sub-diffusion equation. A usual finite difference scheme is applied to discretize the time derivative while cubic trigonometric B-spline is utilized as an interpolating function in the space with the help of Riemann–Liouville operator [12,13,16]. The results obtained by the present method are compared with those obtained in [16] and it can be concluded that the cubic trigonometric B-spline provides better accuracy. The stability analysis of the said technique is also investigated and it is shown to be unconditionally stable.

1.4. Outlines of present paper

This paper is structured as follows: A cubic trigonometric B-spline collocation approach is presented in Section 2. Approximate solution of the fractional sub-diffusion problem is discussed in Section 3. The von Neumann approach is used to investigate the stability of the method in Section 4. One test problem is considered in Section 5 to show the feasibility of the proposed method. Finally, in Section 6, the conclusion of this study is given.

2. Cubic trigonometric B-spline technique

This section presents the cubic trigonometric B-splines collocation method (CuTBS) for the numerical solution of fractional diffusion Eq. (1.1). We assume that the interval $[a, b]$ is divided into N subintervals of uniform length h by inserting the knots x_m such that $a = x_0 < x_1 < \cdots x_N = b$, where $h = x_{m+1} - x_m$, $m = 0, 1, \ldots, N$. Now the cubic B-splines $TB_j(x)$, which are twice continuously differentiable at the knots x_j over the interval $[a, b]$, are defined by [17–21]

$$TB_j(x) = \frac{1}{w}\begin{cases} p^3(x_i) & x \in [x_i, x_{i+1}] \\ p(x_i)(p(x_i)q(x_{i+2}) + q(x_{i+3})p(x_{i+1})) + q(x_{i+4})p^2(x_{i+1}), & x \in [x_{i+1}, x_{i+2}] \\ q(x_{i+4})(p(x_{i+1})q(x_{i+3}) + q(x_{i+4})p(xi+2)) + p(x_i)q^2(x_{i+3}), & x \in [x_{i+2}, x_{i+3}] \\ q^3(x_{i+4}), & x \in [x_{i+1}, x_{i+2}], \end{cases} \tag{2.1}$$

where

$$p(x_i) = \sin\left(\frac{x - x_i}{2}\right), q(x_i) = \sin\left(\frac{x_i - x}{2}\right), w = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right). \tag{2.2}$$

The approximation u_j^k at the point (x_j, t_k) over the subinterval $[x_j, x_{j+1}]$ can be defined as:

$$u(x_j, t_k) = u_j^k = \sum_{j=i-1}^{i+1} \delta_j^k(t)TB_j(x), \tag{2.3}$$