Applied Mathematics and Computation 292 (2017) 1–8

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Time decay rate of weak solutions to the generalized MHD equations in $\mathbb{R}^2$ ☆

Caidi Zhao*, Bei Li

Department of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang, 325035, PR China

A R T I C L E   I N F O

MSC:
76A10
76D03
35Q35

Keywords:
Generalized MHD equations
Fractional dissipation
Time decay rate

A B S T R A C T

This paper studies the time decay rate of weak solutions to the following two-dimensional magnetohydrodynamics (MHD) equations with fractional dissipations

$$\partial_t u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla p = -(-\triangle)^{\alpha}u,$$
$$\partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u = -(-\triangle)^{\beta}b.$$

The motivation is to understand how the parameters α and β affect the decay rate of its solutions. The authors use the Fourier splitting method of Schonbek to prove that the solutions have the following decay rate

$$\|u(x,t)\|^2 + \|b(x,t)\|^2 \leqslant c(1+t)^{1-2/\gamma}, \text{ for large enough } t,$$

where $\alpha, \beta \in [1, 2)$ and $\gamma = \max\{\alpha, \beta\}$.

1. Introduction

In this paper, we study the following equations

$$\partial_t u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla p = -(-\triangle)^{\alpha}u, \tag{1.1}$$

$$\partial_t b + (u\cdot\nabla)b - (b\cdot\nabla)u = -(-\triangle)^{\beta}b, \tag{1.2}$$

$$\nabla\cdot u = \nabla\cdot b = 0, \tag{1.3}$$

$$u(x,0) = u_0, \quad b(x,0) = b_0, \tag{1.4}$$

where the velocity field $u = u(x,t) \in \mathbb{R}^2$, the magnetic field $b = b(x,t) \in \mathbb{R}^2$ and the total pressure $p(x,t) \in \mathbb{R}$ are functions of $x \in \mathbb{R}^2$ and $t \geq 0$. Here α and β are the parameters of the fractional dissipations corresponding to the velocity filed and magnetic field, respectively. A fractional power of the Laplace transform $(-\Delta)^{\lambda}$ is defined through the Fourier transform

$$\widehat{(-\Delta)^{\lambda} f}(\xi) \triangleq |\xi|^{2\lambda}\hat{f}(\xi).$$

☆ Supported by NSFC of China (No. 11271290) and GSPT of Zhejiang Province (No. 2014R424062).

* Corresponding author..
E-mail address: zhaocaidi2013@163.com (C. Zhao).

http://dx.doi.org/10.1016/j.amc.2016.07.028
0096-3003/

Eqs. (1.1) and (1.2) describe the large eddy simulation model for the turbulent flow of a magnetofluid (see [5]). They are of interest for various reasons. For example, they include some known equations, say standard MHD equations ($\alpha = \beta = 1$) and Navier–Stokes (NS) equations ($\alpha = \beta = 1, b = 0$). It is therefore reasonable to call Eqs. (1.1) and (1.2) generalized MHD equations. Moreover, they have similar scaling properties and energy estimate as the NS equations and MHD equations.

There are some papers studying the MHD equations or its related versions. First, the solvability of the MHD equations was investigated in the beginning of 1960s [19]. For example, Wu studied systematically the well-posedness of the two-dimensional (2D) and 3D generalized MHD equations in [21]. Fan et al. investigated the global Cauchy problem of 2D generalized MHD equations in [9]. Zhou and his group studied the regularity criteria for the MHD equations in [10,11,13,29,30,33,34]. The existence and dimension estimation of the global attractor for the MHD equations were verified in [4,5,22]. In addition, the Hall-MHD equations were studied by Wan and Zhou [20]. However, as far as we know, there is no reference investigating the time decay rate of solutions to the generalized MHD equations.

The aim of this paper is to prove an upper bound of the time decay rate of weak solutions to the generalized MHD Eqs. (1.1)–(1.4) in $\mathbb{R}^2$. The motivation is to understand how the parameters α and β affect the decay rate of its solutions. Our main approaches are Fourier splitting method of Schonbek (see [14]) and the Gronwall inequality. Within this paper, we take $\alpha, \beta \in [1, 2)$ and prove that the solutions have the following decay rate

$$\|u(x,t)\|^2 + \|b(x,t)\|^2 \leqslant c(1+t)^{1-2/\gamma}, \text{ for large enough } t, \tag{1.5}$$

hereinafter $\gamma = \max\{\alpha, \beta\}$. Here we want to point out that Wu proved in [21]: For given $u_0, b_0 \in \mathbb{L}^2(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, if $\alpha, \beta \geq 1$, then Eqs. (1.1)–(1.4) possess a global weak solution. We note that if $\alpha, \beta \geq 2$, then the decay rate (1.5) is trivial. This is the reason we take $\alpha, \beta \in [1, 2)$ within this paper.

The Fourier splitting method was built up by Schonbek in [14–17], where the upper and lower bounds of decay rate for the Leray-Hopf solutions of the NS equations were subtly proved. Later, the Fourier splitting method has been well extended to investigate the decay rate for the solutions of partial differential equations from mathematical physics. For example, one can see Bjorland and Schonbek [2], Brandolese and Schonbek[3], Dai, Qing and Schonbek [6], Dong and Chen [7], Dong and Li [8], Niche and Schonbek [12], Schonbek and Wiegner[18], Zhang [23–26], Zhao et al. [27,28], Zhou [31,32], etc.

Different to the NS equations studied in [14], Eqs. (1.1) and (1.2) contain the Maxwell's equations (which rules the magnetic field) and the NS equations (which governs the fluid motion), what is more, Eqs. (1.1) and (1.2) contain the fractional dissipative terms $-(-\triangle)^{\alpha} u$ and $-(-\triangle)^{\beta} b$, respectively. Dissipation corresponding to a fractional power of Laplacian in principle arises from modeling real physical phenomena, but our motivation for studying Eqs. (1.1)–(1.4) is mainly mathematical and the goal is to understand how the parameters α and β affect the decay rate of its solutions.

2. Preliminaries

Throughout this paper, we use c to denote the generic constant that can take different values in different places. $\mathbb{L}^p(\mathbb{R}^2) = L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$ represents the 2D vector Lebesgue space with norm $\|\cdot\|_p$, particularly, $\|\cdot\|_2 = \|\cdot\|$. $\mathbb{W}^{m,p}(\mathbb{R}^2)$ stands for the 2D vector Sobolev space $\{\phi = (\phi_1, \phi_2) \in \mathbb{L}^p(\mathbb{R}^2) \big| D^\eta \phi \in \mathbb{L}^p(\mathbb{R}^2), |\eta| \leqslant m\}$, with norm (see [1])

$$\|\phi\|_{m,p} \triangleq \left(\sum_{k=1}^{2} |\phi_k|_{m,p}^p\right)^{1/p}, \text{ where } |\phi_k|_{m,p} \triangleq \left(\int_{\mathbb{R}^2}\left(\sum_{|\eta|\leqslant m} |D^\eta \phi_k|^p\right)\mathrm{d}x\right)^{1/p}.$$

Especially, we denote $\mathbb{H}^m(\mathbb{R}^2) = \mathbb{W}^{m,2}(\mathbb{R}^2)$ and by $\mathbb{H}_0^m(\mathbb{R}^2)$ the closure of $\{\varphi \in (C_0^\infty(\mathbb{R}^2))^2\}$ with respect to $\mathbb{H}^m(\mathbb{R}^2)$ norm. Note that $\mathbb{H}^m(\mathbb{R}^2) = \mathbb{H}_0^m(\mathbb{R}^2)$. In addition, the space $\mathbb{H}^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, consists of functions f satisfying

$$\|f\|_{\mathbb{H}^s}^2 \triangleq \int_{\mathbb{R}^2} \left(1+|\xi|^2\right)^s \left|\hat{f}(\xi)\right|^2 \mathrm{d}\xi < \infty.$$

We next specify the definition of weak solutions to Eqs. (1.1)–(1.4). Let $T > 0$ be arbitrarily fixed. As usual, we write $(-\Delta)^{1/2}$ as Λ.

Definition 2.1. A measurable vector pair (u, b) is called a weak solution to the generalized MHD Eqs. (1.1)–(1.4), if (u, b) satisfies

$$u \in L^\infty([0,T]; \mathbb{L}^2(\mathbb{R}^2)) \cap L^2([0,T]; \mathbb{H}^\alpha(\mathbb{R}^2)), \quad b \in L^\infty([0,T]; \mathbb{L}^2(\mathbb{R}^2)) \cap L^2([0,T]; \mathbb{H}^\beta(\mathbb{R}^2)),$$

and

$$\int_0^T \int_{\mathbb{R}^2} \Big[u \cdot (\Lambda^{2\alpha} - u \cdot \nabla)\psi + b \cdot \nabla\psi \cdot b - p(\nabla \cdot \psi)\Big]\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \psi(u_0 - u(t))\mathrm{d}x,$$

$$\int_0^T \int_{\mathbb{R}^2} \Big[b \cdot (\Lambda^{2\beta} - u \cdot \nabla)\psi + b \cdot \nabla\psi \cdot u\Big]\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}^2} \psi(b_0 - b(t))\mathrm{d}x,$$